Lecture 6: Recursive Preferences

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April 14, 2023

Overview

- Epstein and Zin (1989 JPE, 1991 Ecta) introduced a class of preferences which allow to break the link between risk aversion and intertemporal substitution.
- These preferences have proved very useful in applied work in asset pricing, portfolio choice, and macroeconomics

Value function

- The standard expected utility time-separable preferences are defined as

$$V_t = \sum_{s=0}^{\infty} \beta^s E_t u\left(c_{t+s}\right)$$

Alternatively can write it as

$$\begin{aligned} V_t &= (1-\beta) \sum_{s=0}^{\infty} \beta^s E_t u\left(c_{t+s}\right) \\ &= (1-\beta) u\left(c_t\right) + \beta E_t\left(V_{t+1}\right) \end{aligned}$$

- V_t is known as value function or lifetime utility

EZ Preferences

- EZ preferences generalize this: they are defined recursively over current (known) consumption and a certainty equivalent $H_t(V_{t+1})$ of tomorrow's utility V_{t+1}:

  $$V_t = F(c_t, H_t(V_{t+1}))$$

  where

  $$H_t(V_{t+1}) = G^{-1}(E_t G(V_{t+1}))$$

  with F and G increasing and concave, and F homogeneous of degree one

EZ Preferences

- **Observation**: F is homogeneous of degree one if

$$F(tX, tY) = tF(X, Y), \text{ for } t > 0$$

and

$$F = X \cdot F'_X + Y \cdot F'_Y$$

Also known as Euler's theorem

- Note that

$$H_t\left(V_{t+1}\right) = G^{-1}\left(E_t G\left(V_{t+1}\right)\right)$$

$$H_t(V_{t+1}) = V_{t+1}$$

if there is no uncertainty on V_{t+1}

- The more concave G is, and the more uncertain V_{t+1} is, the lower is $H_t(V_{t+1})$

Functional forms

- Most of the literature considers simple functional forms for F and G
  - $$G\left(x\right) \equiv \frac{x^{1-\alpha}}{1-\alpha}, \alpha > 0$$
  - $$F\left(c, z\right) \equiv \left((1-\beta)c^{1-\gamma} + \beta z^{1-\gamma}\right)^{\frac{1}{1-\gamma}}, \gamma > 0$$
- For example

$$V_t \equiv \left((1-\beta)c_t^{1-\gamma} + \beta\left(E_t\left(V_{t+1}^{1-\alpha}\right)\right)^{\frac{1-\gamma}{1-\alpha}}\right)^{\frac{1}{1-\gamma}}$$

Functional forms

Proposition: If c_t is deterministic we have the **standard** time-separable expected discounted utility with discount factor β, and IES $=1/\gamma$ and risk aversion $=\gamma$. Also, when $\alpha = 0$ have the **standard** utility function.
Proof: Given

$$V_t \equiv \left((1-\beta) c_t^{1-\gamma} + \beta \left(E_t \left(V_{t+1}^{1-\alpha} \right) \right)^{\frac{1-\gamma}{1-\alpha}} \right)^{\frac{1}{1-\gamma}}$$

If c_t is deterministic then

$$(V_t)^{1-\gamma} = (1-\beta) c_t^{1-\gamma} + \beta (V_{t+1})^{1-\gamma} = (1-\beta) \sum_{s=0}^{\infty} \beta^s c_{t+s}^{1-\gamma},$$

when $\alpha = 0$ iterate forward to get

$$(V_t)^{1-\gamma} = (1-\beta) c_t^{1-\gamma} + \beta E_t (V_{t+1})^{1-\gamma} = (1-\beta) \sum_{s=0}^{\infty} \beta^s E_t u(c_{t+s}).$$

Limits

If

- $\gamma = 1$

$$F(c, z) = c^{1-\beta} z^{\beta}$$

- $\alpha = 1$

$$G(x) = \log x$$

Thus:

- $\alpha > 0$

$$H_t(V_{t+1}) = \left[E_t \left(V_{t+1} \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}}$$

- $\alpha = 1$

$$H_t(V_{t+1}) = \exp\left(E_t \log(V_{t+1}) \right)$$

Proof

Define

$$F(c,z) = \left((1-\beta)c^{1-\gamma} + \beta z^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

divide and multiply by c

$$\begin{aligned} F(c,z) &= c\left((1-\beta) + \beta x^{1-\gamma}\right)^{\frac{1}{1-\gamma}} \\ &= cf(x) \end{aligned}$$

where

$$x = z/c \text{ and } f(x) = \left(1-\beta+\beta x^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

so

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{1}{1-\gamma}\frac{\left(1-\beta+\beta x^{1-\gamma}\right)^{\frac{1}{1-\gamma}-1}(1-\gamma)\beta x^{-\gamma}}{\left(1-\beta+\beta x^{1-\gamma}\right)^{\frac{1}{1-\gamma}}} \\ &= \frac{\beta x^{-\gamma}}{\left(1-\beta+\beta x^{1-\gamma}\right)} \end{aligned}$$

Proof

- And

$$\lim_{\gamma\to 1}\frac{f'(x)}{f(x)} = \lim_{\gamma\to 1}\frac{\beta x^{-\gamma}}{\left(1-\beta+\beta x^{1-\gamma}\right)}$$
$$= \lim_{\gamma\to 1}\frac{\beta}{(1-\beta)\,x^{\gamma}+\beta x^{1}} = \frac{\beta}{x}$$

- Since f is continuous then

$$\lim_{\gamma\to 1} f(x) = x^{\beta}$$

or

$$F(c,z) = cf(x) = c\,(z/c)^{\beta} = c^{1-\beta}z^{\beta}$$

Cobb-Douglas function

Risk Aversion vs IES

- In general α is the relative risk aversion coefficient for static gambles and γ is the inverse of the intertemporal elasticity of substitution for deterministic variations
- Suppose consumption is c today and consumption tomorrow is uncertain: $\{c_L, c, c, ...\}$ or $\{c_H, c, c, ...\}$, each has prob. 0.5
- Lifetime utility today

$$V_t = F\left(c, G^{-1}\left(0.5G\left(V_L\right) + 0.5G\left(V_H\right)\right)\right)$$

where

$$V_L = F\left(c_L, \overline{c}\right), V_H = F\left(c_H, \overline{c}\right)$$

- Curvature of G determines how adverse you are to the uncertainty.
  - If G is linear you only care about the expected value
  - If not, it is the certainty equivalent:

$$G\left(\widehat{V}\right) = 0.5G\left(V_L\right) + 0.5G\left(V_H\right)$$

Special Case: Deterministic consumption

- If consumption is deterministic: we have the usual standard time-separable expected discounted utility with discount factor β and $IES = 1/\gamma$, risk aversion $= \alpha$.
- Proof: Without uncertainty, then $H_t(V_{t+1}) = G^{-1}\left(E_t G\left(V_{t+1}\right)\right) = V_{t+1}$ and

$$V_t = F\left(c_t, H_t\left(V_{t+1}\right)\right)$$

With a CES functional form for F, we recover *CRRA* preferences:

$$V_t = \left[(1-\beta)c_t^{1-\gamma} + \beta\left(V_{t+1}\right)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

$$U_t = (1-\beta)c_t^{1-\gamma} + \beta U_{t+1} = (1-\beta)\sum_{s=0}^{\infty}\beta^s c_{t+s}^{1-\gamma}$$

where

$$U_t = \left(V_t\right)^{1-\gamma}.$$

Special Case: RRA=1/IES

- if $\alpha = \gamma$, then the formula

$$V_t \equiv \left((1-\beta) c_t^{1-\gamma} + \beta \left(E_t \left(V_{t+1}^{1-\alpha} \right) \right)^{\frac{1-\gamma}{1-\alpha}} \right)^{\frac{1}{1-\gamma}}$$

simplifies to

$$(V_t)^{1-\gamma} \equiv (1-\beta) c_t^{1-\gamma} + \beta \left(E_t V_{t+1}^{1-\gamma} \right)$$

- Define

$$U_t = V_t^{1-\gamma}$$

then

$$U_t = (1-\beta) c_t^{1-\gamma} + \beta E_t (U_{t+1}),$$

is the expected utility

Simple example with two lotteries

Lotteries:

- lottery A pays in each period $t = 1, 2, ...$ either c_h or c_l, with probability 0.5 and the outcome is iid across periods;
- lottery B pays starting at $t = 1$ either c_h at all future dates for sure, or c_l at all future dates for sure; there is a single draw at time $t = 1$
- With expected utility, you are indifferent between these lotteries, but with EZ lottery B is preferred iff $\alpha > \gamma$.
- In general, early resolution of uncertainty is preferred if and only if $\alpha > \gamma$ i.e. risk aversion $> \frac{1}{IES}$
- This is another way to motivate these preferences, since early resolution seems intuitively preferable.

Resolution of uncertainty

$$V_t \equiv \left((1-\beta) c_t^{1-\gamma} + \beta \left(E_t \left(V_{t+1}^{1-\alpha} \right) \right)^{\frac{1-\gamma}{1-\alpha}} \right)^{\frac{1}{1-\gamma}}$$

- For lottery B, the utility once you know your consumption is either c_h, or c_l forever,

$$V_h = F\left(c_h, V_h\right) = \left((1-\beta) c_h^{1-\gamma} + \beta V_h^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

or

$$V_l = F\left(c_l, V_l\right) = \left((1-\beta) c_l^{1-\gamma} + \beta V_l^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

- The certainty equivalent before playing the lottery is

$$G^{-1}\left(0.5 G\left(c_h\right) + 0.5 G\left(c_l\right)\right) = \left(0.5 c_h^{1-\alpha} + 0.5 c_l^{1-\alpha}\right)^{\frac{1}{1-\alpha}}$$

Resolution of uncertainty

- Given

$$W_t \equiv \left((1-\beta) c_t^{1-\gamma} + \beta \left(E_t \left(W_{t+1}^{1-\alpha} \right) \right)^{\frac{1-\gamma}{1-\alpha}} \right)^{\frac{1}{1-\gamma}}$$

- For lottery A, the values satisfy

$$W_h^{1-\gamma} = (1-\beta) c_h^{1-\gamma} + \beta \left(0.5 W_h^{1-\alpha} + 0.5 W_l^{1-\alpha} \right)^{\frac{1-\gamma}{1-\alpha}}$$

$$W_l^{1-\gamma} = (1-\beta) c_l^{1-\gamma} + \beta \left(0.5 W_h^{1-\alpha} + 0.5 W_l^{1-\alpha} \right)^{\frac{1-\gamma}{1-\alpha}}$$

Resolution of uncertainty

- We want to compare

$$G^{-1}\left(0.5G\left(c_h\right)+0.5G\left(c_l\right)\right)$$

with

$$G^{-1}\left(0.5G\left(W_h\right)+0.5G\left(W_l\right)\right)$$

- **notice** that function

$$x^{\frac{1-\gamma}{1-\alpha}}$$

is concave if $1-\gamma<1-\alpha$, i.e. $\gamma>\alpha$, and convex otherwise. As a result, if $\gamma>\alpha$

$$\begin{aligned}\left(0.5W_h^{1-\alpha}+0.5W_l^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}} &> 0.5\left(W_h^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}}+0.5\left(W_l^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}} \\ &= 0.5W_h^{1-\gamma}+0.5W_l^{1-\gamma}\end{aligned}$$

Resolution of uncertainty

Since

$$W_h^{1-\gamma} = (1-\beta)c_h^{1-\gamma} + \beta\left(0.5W_h^{1-\alpha} + 0.5W_l^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}}$$

and

$$W_l^{1-\gamma} = (1-\beta)c_l^{1-\gamma} + \beta\left(0.5W_h^{1-\alpha} + 0.5W_l^{1-\alpha}\right)^{\frac{1-\gamma}{1-\alpha}}$$

Then

$$W_h^{1-\gamma} > (1-\beta)c_h^{1-\gamma} + \beta\left(0.5W_h^{1-\gamma} + 0.5W_l^{1-\gamma}\right)$$

and

$$W_l^{1-\gamma} > (1-\beta)c_l^{1-\gamma} + \beta\left(0.5W_h^{1-\gamma} + 0.5W_l^{1-\gamma}\right)$$

Resolution of uncertainty

- Multiplying both equations by 0.5 and summing them up

$$(1-\beta)\left(0.5W_h^{1-\gamma}+0.5W_l^{1-\gamma}\right) > (1-\beta)\left(0.5c_h^{1-\gamma}+0.5c_l^{1-\gamma}\right)$$

- These results imply that if $\gamma > \alpha$ then

$$0.5W_h^{1-\gamma}+0.5W_l^{1-\gamma} > 0.5c_h^{1-\gamma}+0.5c_l^{1-\gamma}$$

- In this case the certainty equivalent of lottery A is higher than the certainty equivalent of lottery B and agents prefer late to early resolution of uncertainty.

Stochastic Discount Factor

- The stochastic discount factor with these preferences turns out to be slightly different:

$$m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left[\frac{V_{t+1}}{\left(E_t V_{t+1}^{1-\alpha} \right)^{\frac{1}{1-\alpha}}} \right]^{\gamma - \alpha}$$

- The first term is familiar. The second term is next period's value (lifetime utility) relative to its certainty equivalent.

Stochastic Discount Factor

- **Proof:** Since F is homogenous of degree one,

$$V_t = \left((1-\beta) c_t^{1-\gamma} + \beta \left(H_t \left(V_{t+1} \right) \right)^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

where

$$H_t(V_{t+1}) = \left[E_t \left(V_{t+1} \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}}$$

Euler's theorem implies

$$V_t = \frac{\partial V_t}{\partial c_t} c_t + E_t \frac{\partial V_t}{\partial V_{t+1}} V_{t+1}$$

$$\frac{\partial V_t}{\partial c_t} = (1-\beta) V_t^{\gamma} c_t^{-\gamma}$$

and

$$\frac{\partial V_t}{\partial V_{t+1}} = \frac{\partial V_t}{\partial H_t \left(V_{t+1} \right)} \frac{\partial H_t \left(V_{t+1} \right)}{\partial V_{t+1}}$$

Stochastic Discount Factor

- Since

$$V_t = \left((1-\beta) c_t^{1-\gamma} + \beta \left(H_t \left(V_{t+1} \right) \right)^{1-\gamma} \right)^{\frac{1}{1-\gamma}}$$

where

$$H_t(V_{t+1}) = \left[E_t \left(V_{t+1} \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}}$$

then

$$\frac{\partial V_t}{\partial H_t \left(V_{t+1} \right)} = \beta V_t^{\gamma} \left(H_t \left(V_{t+1} \right) \right)^{-\gamma}$$

and

$$\frac{\partial H_t \left(V_{t+1} \right)}{\partial V_{t+1}} = \left(H_t \left(V_{t+1} \right) \right)^{\alpha} V_{t+1}^{-\alpha}$$

this implies

$$\frac{\partial V_t}{\partial V_{t+1}} = \beta V_t^{\gamma} \left(H_t \left(V_{t+1} \right) \right)^{\alpha-\gamma} V_{t+1}^{-\alpha}$$

Stochastic Discount Factor

- Divide the value function

$$V_t = \frac{\partial V_t}{\partial c_t} c_t + E_t \frac{\partial V_t}{\partial V_{t+1}} V_{t+1}$$

by $\frac{\partial V_t}{\partial c_t}$

$$\frac{V_t}{\frac{\partial V_t}{\partial c_t}} = c_t + E_t \frac{\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}} \frac{V_{t+1}}{\frac{\partial V_{t+1}}{\partial c_{t+1}}}$$

- Let W_t be the wealth, then by definition

$$W_t = c_t + E_t m_{t+1} W_{t+1}$$

This implies that

$$W_t = \frac{V_t}{\frac{\partial V_t}{\partial c_t}} = \frac{V_t}{(1-\beta) V_t^{\gamma} c_t^{-\gamma}} = \frac{V_t^{1-\gamma} c_t^{\gamma}}{(1-\beta)}$$

Stochastic Discount Factor

- And the stochastic discount factor

$$
\begin{aligned}
m_{t+1} &= \frac{\frac{\partial V_t}{\partial V_{t+1}} \frac{\partial V_{t+1}}{\partial c_{t+1}}}{\frac{\partial V_t}{\partial c_t}} = \frac{\left[\beta V_t^{\gamma} (H_t(V_{t+1}))^{\alpha-\gamma} V_{t+1}^{-\alpha}\right]\left[(1-\beta) V_{t+1}^{\gamma} c_{t+1}^{-\gamma}\right]}{(1-\beta) V_t^{\gamma} c_t^{-\gamma}} \\
&= \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \frac{\left(H_t\left(V_{t+1}\right)\right)^{\alpha-\gamma}}{V_{t+1}^{\alpha-\gamma}} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left(\frac{\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}}{V_{t+1}} \right)^{\alpha-\gamma}
\end{aligned}
$$

Thus

$$
m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left[\frac{V_{t+1}}{\left(E_t V_{t+1}^{1-\alpha}\right)^{\frac{1}{1-\alpha}}} \right]^{\gamma-\alpha}
$$

Stochastic Discount Factor

- Define the cumulative dividend return on wealth

$$W_{t+1} = R_{m,t+1}(W_t - c_t)$$

or

$$\frac{V_{t+1}^{1-\gamma} c_{t+1}^{\gamma}}{(1-\beta)} = R_{m,t+1}\left(\frac{V_t^{1-\gamma} c_t^{\gamma}}{(1-\beta)} - c_t\right)$$

hence

$$\begin{aligned} R_{m,t+1} &= \frac{V_{t+1}^{1-\gamma} c_{t+1}^{\gamma}}{V_t^{1-\gamma} c_t^{\gamma} - (1-\beta) c_t} \\ &= \frac{c_{t+1}^{\gamma}}{c_t^{\gamma}} \frac{V_{t+1}^{1-\gamma}}{V_t^{1-\gamma} - (1-\beta) c_t^{1-\gamma}} \end{aligned}$$

Stochastic Discount Factor

- Now use the fact that

$$V_t^{1-\gamma} \equiv (1-\beta) c_t^{1-\gamma} + \beta \left(H_t \left(V_{t+1} \right) \right)^{1-\gamma}$$

to replace in the equation:

$$\begin{aligned} R_{m,t+1} &= \frac{c_{t+1}^{\gamma}}{c_t^{\gamma}} \frac{V_{t+1}^{1-\gamma}}{V_t^{1-\gamma} - (1-\beta) c_t^{1-\gamma}} \\ &= \frac{c_{t+1}^{\gamma}}{\beta c_t^{\gamma}} \left(\frac{V_{t+1}}{H_t \left(V_{t+1} \right)} \right)^{1-\gamma} \end{aligned}$$

Use this equation to solve for the value function relative to its certainty equivalent:

$$\frac{V_{t+1}}{H_t \left(V_{t+1} \right)} = \left[\beta \frac{c_t^{-\gamma}}{c_{t+1}^{-\gamma}} R_{m,t+1} \right]^{\frac{1}{1-\gamma}}$$

Stochastic Discount Factor

- $$m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left(\frac{V_{t+1}}{H_t\left(V_{t+1}\right)} \right)^{\gamma-\alpha}$$

  can be written using the expression for

  $$\frac{V_{t+1}}{H_t\left(V_{t+1}\right)} = \left[\beta \frac{c_t^{-\gamma}}{c_{t+1}^{-\gamma}} R_{m,t+1} \right]^{\frac{1}{1-\gamma}}$$

  as

  $$\begin{aligned} m_{t+1} &= \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} \left[\beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}} R_{m,t+1} \right]^{\frac{\gamma-\alpha}{1-\gamma}} \\ &= \beta^{\theta} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma\theta} R_{m,t+1}^{\theta-1} \end{aligned}$$

  where

  $$\theta = \frac{1-\alpha}{1-\gamma}$$

Stochastic Discount Factor

- Now take logs

$$m_{t+1} = \beta^{\theta} \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma\theta} R_{m,t+1}^{\theta-1}$$

to obtain

$$\log m_{t+1} = \theta \log \beta - \gamma\theta \log \left(\frac{c_{t+1}}{c_t} \right) + (\theta - 1) \log R_{m,t+1}$$

- In lecture 2 (where $\alpha = \gamma$)

$$m_{t+1} = \beta \frac{c_{t+1}^{-\gamma}}{c_t^{-\gamma}}$$

$$\log m_{t+1} = \log \beta - \gamma \log \left(\frac{c_{t+1}}{c_t} \right)$$

Risk free rate

- Assume that both $\log\left(\frac{c_{t+1}}{c_t}\right)$ is normal distributed and $\log\left(R_{m,t+1}\right)$ is normal distributed and they are independent distributed.
- **Remember**: If z is normal distributed then $\exp(z)$ is lognormal. Also $E\exp(z) = \exp(Ez + 0.5\sigma^2(z))$.
- Let $\Delta c_{t+1} = \log\left(\frac{c_{t+1}}{c_t}\right)$ and $\log\left(R_{m,t+1}\right) = r_{m,t+1}$.
- Then $\exp(\Delta c_{t+1})$ and $\exp(r_{m,t+1})$ are lognormal distributed.

$$\left(R^f_{t+1}\right)^{-1} = E_t\left(m_{t+1}\right)$$

$$\left(R^f_{t+1}\right)^{-1} = E_t \exp\left(\log\left(\beta^\theta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma\theta} R^{\theta-1}_{m,t+1}\right)\right)$$

$$\left(R^f_{t+1}\right)^{-1} = e^{\log\beta^\theta} E_t e^{-\gamma\theta(\Delta c_{t+1})} E_t e^{-(1-\theta) r_{m,t+1}}$$

using the fact that for independent variables $Exy = ExEy$.

Risk free rate

$$\left(R^f_{t+1}\right)^{-1} = e^{\log \beta^{\theta}} e^{-\gamma\theta E_t(\Delta c_{t+1}) + \frac{1}{2}(\gamma\theta)^2 \sigma^2(\Delta c_{t+1})} e^{-(1-\theta)E_t r_{m,t+1} + \frac{1}{2}(1-\theta)^2 \sigma^2(r_{m,t+1})}$$

- taking logarithms

$$\begin{aligned} r^f_{t+1} &= -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) - \frac{1}{2}\left(\gamma\theta\right)^2 \sigma^2\left(\Delta c_{t+1}\right) \\ &\quad + \left(1-\theta\right) E_t r_{m,t+1} - \frac{1}{2}\left(1-\theta\right)^2 \sigma^2\left(r_{m,t+1}\right) \end{aligned}$$

- if $r_{m,t+1}$ and Δc_{t+1} are jointly lognormal distributed (not independent) then there is an additional term

$$r^f_{t+1} = ... - \gamma\theta\left(1-\theta\right) cov_t(\Delta c_{t+1}, r_{m,t+1})$$

Equity premium

- Assume that consumption growth and asset returns are jointly log-normally distributed like in lecture 2.

$$\left[\begin{array}{c} \Delta c_{t+1} \\ r^i_{t+1} \end{array}\right] \sim N\left(\left[\begin{array}{c} \overline{\Delta c}_{t+1} \\ \overline{r}^i_{t+1} \end{array}\right], \left[\begin{array}{cc} var\left(\Delta c_{t+1}\right) & cov\left(\Delta c_{t+1}, r^i_{t+1}\right) \\ cov\left(\Delta c_{t+1}, r^i_{t+1}\right) & var\left(r^i_{t+1}\right) \end{array}\right]\right)$$

- We established (in lecture 2) that,

$$\overline{r}^i_{t+1} - r^f_{t+1} = -\frac{1}{2} var(r^i_{t+1}) + \gamma cov_t(x_{t+1}, r^i_{t+1})$$

Equity premium

- If $\log\left(R^i_{t+1}\right)$ and $\log m_{t+1}$ are normal distributed
- The Euler equations become

$$1 = \exp\left(E_t \log m_{t+1} + r^f_{t+1} + \frac{1}{2} var(\log m_{t+1})\right)$$

$$1 = \exp\left(E_t \log m_{t+1} + \overline{r}^i_{t+1} + \frac{1}{2} var(\log m_{t+1} + r^i_{t+1})\right)$$

Equity premium

- Take logs and equate these equations:

$$E_t \log m_{t+1} + r^f_{t+1} + \frac{1}{2} var(\log m_{t+1}) = E_t \log m_{t+1} + \overline{r}^i_{t+1} + \frac{1}{2} var(\log m_{t+1} + r^i_{t+1})$$

$$r^f_{t+1} + \frac{1}{2} var(\log m_{t+1}) = \overline{r}^i_{t+1} + \frac{1}{2} var(\log m_{t+1} + r^i_{t+1})$$

$$r^f_{t+1} + \frac{1}{2} var(\log m_{t+1}) = \overline{r}^i_{t+1} + \frac{1}{2} \left[\begin{array}{c} var(\log m_{t+1}) + var(r^i_{t+1}) \\ +2cov(\log m_{t+1}, r^i_{t+1}) \end{array} \right]$$

$$\overline{r}^i_{t+1} - r^f_{t+1} = -\frac{1}{2} var(r^i_{t+1}) - cov(\log m_{t+1}, r^i_{t+1})$$

$$\overline{r}^i_{t+1} + \frac{1}{2} var(r^i_{t+1}) - r^f_{t+1} = -cov(\log m_{t+1}, \log (R^i_{t+1}))$$

Equity premium

$$\log E_t R^i_{t+1} - \log R^f_{t+1} = -cov(\log m_{t+1}, \log (R^i_{t+1}))$$

since

$$\log m_{t+1} = \theta \log \beta - \gamma\theta \log \left(\frac{c_{t+1}}{c_t}\right) + (\theta - 1) \log R_{m,t+1}$$

Then:

$$\log \left(\frac{E_t R^i_{t+1}}{R^f_{t+1}}\right) = \gamma\theta cov_t(\Delta c_{t+1}, r^i_{t+1}) + (1-\theta)\, cov_t(r_{m,t+1}, r^i_{t+1})$$

Conclusion: Epstein-Zin is a linear combination of the CAPM and the CCAPM model

Market return

For the market return we have

$$\log\left(\frac{E_t R_{m,t+1}}{R^f_{t+1}}\right) = \gamma\theta cov_t(\Delta c_{t+1}, r_{m,t+1}) + (1-\theta)\,\sigma_t^2(r_{m,t+1})$$

or

$$\begin{aligned} E_t r_{m,t+1} + \frac{1}{2}\sigma^2\left(r_{m,t+1}\right) - r^f_{t+1} &= \gamma\theta cov_t(\Delta c_{t+1}, r_{m,t+1}) \\ &\quad + (1-\theta)\,\sigma_t^2(r_{m,t+1}) \end{aligned}$$

which we solve for

$$\begin{aligned} (1-\theta)\,E_t r_{m,t+1} &= (1-\theta)\,r^f_{t+1} - \frac{(1-\theta)}{2}\sigma^2\left(r_{m,t+1}\right) + \\ &\quad (1-\theta)\,\gamma\theta cov_t(\Delta c_{t+1}, r_{m,t+1}) + (1-\theta)^2\,\sigma_t^2(r_{m,t+1}) \end{aligned}$$

From a previous "slide" we got the riskless rate

- $$\begin{aligned} r^f_{t+1} &= -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) + \left(1-\theta\right) E_t r_{m,t+1} - \frac{1}{2}\left(\gamma\theta\right)^2 \sigma^2\left(\Delta c \right. \\ &\quad -\frac{1}{2}\left(1-\theta\right)^2 \sigma^2\left(r_{m,t+1}\right) - \gamma\theta\left(1-\theta\right) cov_t\left(\Delta c_{t+1}, r_{m,t+1}\right) \end{aligned}$$

- replacing $\left(1-\theta\right) E_t r_{m,t+1}$

$$\begin{aligned} r^f_{t+1} &= -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) + \left(1-\theta\right) E_t r_{m,t+1} \\ &\quad \left(1-\theta\right) r^f_{t+1} - \frac{\left(1-\theta\right)}{2}\sigma^2\left(r_{m,t+1}\right) + \\ &\quad \left(1-\theta\right)\gamma\theta cov_t\left(\Delta c_{t+1}, r_{m,t+1}\right) + \left(1-\theta\right)^2 \sigma^2_t\left(r_{m,t+1}\right) \\ &\quad -\frac{1}{2}\left(\gamma\theta\right)^2\sigma^2\left(\Delta c_{t+1}\right) \\ &\quad -\frac{1}{2}\left(1-\theta\right)^2\sigma^2\left(r_{m,t+1}\right) - \gamma\theta\left(1-\theta\right) cov_t\left(\Delta c_{t+1}, r_{m,t+1}\right) \end{aligned}$$

Risk free rate

$$\begin{aligned}
\theta r_{t+1}^{f} &= -\theta \log \beta + \gamma\theta E_t\left(\Delta c_{t+1}\right) - \frac{(1-\theta)}{2}\sigma^2\left(r_{m,t+1}\right) + \\
&\quad -\frac{1}{2}\left(\gamma\theta\right)^2\sigma^2\left(\Delta c_{t+1}\right) + \frac{1}{2}\left(1-\theta\right)^2\sigma^2\left(r_{m,t+1}\right)
\end{aligned}$$

$$\begin{aligned}
r_{t+1}^{f} &= -\log \beta + \gamma E_t\left(\Delta c_{t+1}\right) - \frac{(1-\theta)}{2\theta}\sigma^2\left(r_{m,t+1}\right) + \\
&\quad -\frac{1}{2}\gamma^2\theta\sigma^2\left(\Delta c_{t+1}\right) + \frac{1}{2\theta}\left(1-\theta\right)^2\sigma^2\left(r_{m,t+1}\right)
\end{aligned}$$

$$r_{t+1}^{f} = -\log \beta + \gamma E_t\left(\Delta c_{t+1}\right) - \frac{1}{2}\gamma^2\theta\sigma^2\left(\Delta c_{t+1}\right) - \frac{(1-\theta)}{2}\sigma^2\left(r_{m,t+1}\right)$$

Again if $\gamma = \alpha$ then $\theta = 1$ we have the standard risk-free rate equation. If $\alpha > \gamma$ then $\theta < 1$ and the volatility from the market return reduces the real interest rate.

EZ preference and riskless rate

US Historical data

$$
\begin{aligned}
E_t\left(\Delta c_{t+1}\right) &= 0.02, \\
\sigma^2\left(\Delta c_{t+1}\right) &= (0.036)^2 = 0.0013 \\
\sigma^2\left(r_{m,t+1}\right) &= (0.167)^2 = 0.0279
\end{aligned}
$$

with $\beta = 0.98$, $\alpha = 2$ and $\gamma = 0.5$ (which are reasonable) get 1% riskless interest rate

EZ preference and equity premium

Equity premium

$$\log\left(\frac{E_t R_{m,t+1}}{R^f_{t+1}}\right) = \gamma\theta cov_t\left(\Delta c_{t+1}, r_{m,t+1}\right) + \left(1-\theta\right)\sigma_t^2\left(r_{m,t+1}\right)$$

US Historical data

$$\begin{aligned}
E_t\left(\Delta c_{t+1}\right) &= 0.02, \sigma_r = 0.167 \\
\sigma^2\left(\Delta c_{t+1}\right) &= \left(0.036\right)^2 = 0.0013 \\
\sigma^2\left(r_{m,t+1}\right) &= \left(0.167\right)^2 = 0.0279 \\
corr\left(x, r\right) &= 0.4
\end{aligned}$$

with $\beta = 0.98$, $\alpha = 2$ and $\gamma = 0.5$ (which are reasonable) get 7.4% equity premium