1. Consider an exchange economy with 2 periods ($t = 0, 1$) 2 commodities ($h = 1, 2$), 2 consumers ($i = 1, 2$) and 2 states of nature in period 1 ($s = 1, 2$). Consumer's i preferences are represented by expected utility of the form

$$\frac{1}{2}u^i\left(x_1^i(1), x_2^i(1)\right) + \frac{1}{2}u^i\left(x_1^i(2), x_2^i(2)\right)$$

where $x_h^i(s)$ denotes consumption for household i of commodity h in state s in period 1, $h = 1, 2$, $s = 1, 2$, $i = 1, 2$ and

$$u^1\left(x_1^1(s), x_2^1(s)\right) = 4\ln x_1^1(s) + \ln x_2^1(s)$$

$$u^2\left(x_1^2(s), x_2^2(s)\right) = \ln x_1^2(s) + 4\ln x_2^2(s)$$

Consumer 1's endowment is (1,1) in state 1 and (2,2) in state 2. Consumer 2's endowment is (2,2) in state 1 and (1,1) in state 2.

a) Suppose the only markets in this economy are commodity spot markets in each state $s = 1, 2$ in period 1. There are no securities markets. Find the equilibrium (also known as Radner equilibrium) for this market structure. Is the equilibrium allocation optimal?

Answer:

An equilibrium is a price vector $P(s)$ and an allocation $x^i(s)$ such that:

1. $x^i(s) \in B_i(P(s))$ and there is no other $x'^i(s) \in B_i(P(s))$ that gives more utility to consumer i than $x^i(s)$. Where $B_i(P(s)) = \left\{(x^i) : P(s)x^i(s) \leq P(s)e^i(s)\right\}$. The variables $P(s)$, and $e^i(s)$ are exogenous from the consumer's point of view.

2. Markets clear.

$$x^1(s) + x^2(s) = e^1(s) + e^2(s) \text{ for } s = 1, 2$$

$x^i(s)$ is the vector of goods consumed by household i at date 1 when the state is s.

Observation: When the utility function is of the form $x_1^\alpha x_2^{1-\alpha}$ then the optimal expenditure in good 1 is a share α of the total income and on good 2 is $1 - \alpha$.

$$P_1 x_1 = \alpha\left(P_1 e^1 + P_2 e^2\right)$$

Assume that good 1 is the numeraire good and let p be the relative price of good 2. Then

$$x_1 = \alpha\left(e^1 + pe^2\right).$$

Applying this result get the following demand functions for households 1 and 2 when the state of nature is 1:

$$\begin{aligned} x_1^1(1) &= \frac{4}{5}(1+p),\ x_2^1(1) = \frac{1}{5}\left(\frac{1+p}{p}\right), \\ x_1^2(1) &= \frac{1}{5}(2+2p),\ x_2^2(1) = \frac{4}{5}\left(\frac{2+2p}{p}\right), \end{aligned}$$

From the market clearing condition get

$$\frac{4}{5}(1+p)+\frac{1}{5}(2+2p)=3$$

$$\Longrightarrow p(1)=\frac{3}{2}$$

Replacing this price in the above demand functions get

$$\begin{aligned} x_1^1(1) &= \frac{4}{5}(1+\frac{3}{2})=2,\ x_2^1(1)=\frac{1}{5}(\frac{1+\frac{3}{2}}{\frac{3}{2}})=\frac{1}{3}, \\ x_1^2(1) &= \frac{1}{5}(2+2\frac{3}{2})=1,\ x_2^2(1)=\frac{4}{5}(\frac{2+2\frac{3}{2}}{\frac{3}{2}})=\frac{8}{3}, \end{aligned}$$

When the state of nature is 2 the demand functions for households 1 and 2 are

$$\begin{aligned} x_1^1(2) &= \frac{4}{5}(2+2p),\ x_2^1(2)=\frac{1}{5}(\frac{2+2p}{p}), \\ x_1^2(2) &= \frac{1}{5}(1+p),\ x_2^2(2)=\frac{4}{5}(\frac{1+p}{p}), \end{aligned}$$

From the market clearing condition get

$$\frac{4}{5}(2+2p)+\frac{1}{5}(1+p)=3$$

$$\Longrightarrow p(2)=\frac{2}{3}$$

Replacing this price in the above demand functions get

$$\begin{aligned} x_1^1(1) &= \frac{4}{5}(2+2\frac{2}{3})=\frac{8}{3},\ x_2^1(1)=\frac{1}{5}(\frac{2+2\frac{2}{3}}{\frac{2}{3}})=1, \\ x_1^2(1) &= \frac{1}{5}(1+\frac{2}{3})=\frac{1}{3},\ x_2^2(1)=\frac{4}{5}(\frac{1+\frac{2}{3}}{\frac{2}{3}})=2, \end{aligned}$$

The expected utility of this allocation for household 1 is

$$\frac{1}{2}\left(4\ln x_1^1(1)+\ln x_2^1(1)\right)+\frac{1}{2}\left(4\ln x_1^1(2)+\ln x_2^1(2)\right)$$

$$\frac{1}{2}\left(4\ln 2+\ln\frac{1}{3}\right)+\frac{1}{2}\left(4\ln\frac{8}{3}+\ln 1\right)=2.798$$

And the expected utility of this allocation for household 2 is

$$\frac{1}{2}\left(\ln x_1^2(1)+4\ln x_2^2(1)\right)+\frac{1}{2}\left(\ln x_1^2(2)+4\ln x_2^2(2)\right)$$

$$\frac{1}{2}\left(\ln\frac{8}{3}+4\ln 1\right)+\frac{1}{2}\left(\ln\frac{1}{3}+4\ln 2\right)=2.798$$

The Pareto allocation is the one that solves the problem:

$$\max_{\{x^i\}} \sum_i \varpi_i E u^i\left(x^i\right)$$

subject to

$$x_h^1(s) + x_h^2(s) = e_h^1(s) + e_h^2(s), \text{ for } h = 1,2, \ s = 1,2$$

where the ϖ_i is the weight of household i in the social planner's objective function. Let $\lambda_h(s)$ be the Lagrangian multipliers of the constraints. The first order conditions are:

$$\varpi_i \pi(s) u_{x_h}^i(s) = \lambda_h(s), \text{ for } h = 1,2 \ , s = 1,2, \ i = 1,2$$

Assuming $\varpi_1 = \varpi_2$, this implies:

$$\frac{\pi(1) u_{x_h}^1(1)}{\pi(2) u_{x_h}^1(2)} = \frac{\pi(1) u_{x_h}^2(1)}{\pi(2) u_{x_h}^2(2)} = \frac{\lambda_h(1)}{\lambda_h(2)}, \text{for } h = 1,2$$

where $\pi(1) = \pi(2) = 0.5$, and

$$\frac{u_{x_1}^1(s)}{u_{x_2}^1(s)} = \frac{u_{x_1}^2(s)}{u_{x_2}^2(s)} = \frac{\lambda_1(s)}{\lambda_2(s)}, \text{ for } s = 1,2$$

Conclusion: equal MRS across goods and states of nature for the households.

In our case:

$$\frac{x_1^1(1)}{x_1^1(2)} = \frac{x_1^2(1)}{x_1^2(2)}$$

$$\frac{x_2^1(1)}{x_2^1(2)} = \frac{x_2^2(1)}{x_2^2(2)}$$

and

$$\frac{4x_2^1(1)}{x_1^1(1)} = \frac{x_2^2(1)}{4x_1^1(1)}$$

$$\frac{4x_2^1(2)}{x_1^1(2)} = \frac{x_2^2(2)}{4x_1^1(2)}$$

plus the 4 feasibility constraints determine the 8 quantities: $\{x_1^1(1), \ x_2^1(1), \ x_1^2(1), x_2^2(1), x_1^1(2), x_2^1(2), x_1^2(2), x_2^2(2)\}$. The solution is $\{\frac{24}{10}, \frac{6}{10}, \frac{6}{10}, \frac{24}{10}, \frac{24}{10}, \frac{6}{10}, \frac{6}{10}, \frac{24}{10}\}$.

The expected utility of the social planner's allocation for household 1

$$\frac{1}{2}\left(4\ln x_1^1(1) + \ln x_2^1(1)\right) + \frac{1}{2}\left(4\ln x_1^1(2) + \ln x_2^1(2)\right)$$

$$\frac{1}{2}\left(4\ln\frac{24}{10} + \ln\frac{6}{10}\right) + \frac{1}{2}\left(4\ln\frac{24}{10} + \ln\frac{6}{10}\right) = 2.99$$

And the expected utility of of the social planner's allocation for household 2 is

$$\frac{1}{2}\left(\ln x_1^2(1) + 4\ln x_2^2(1)\right) + \frac{1}{2}\left(\ln x_1^2(2) + 4\ln x_2^2(2)\right)$$

$$\frac{1}{2}\left(\ln\frac{6}{10} + 4\ln\frac{24}{10}\right) + \frac{1}{2}\left(\ln\frac{6}{10} + 4\ln\frac{24}{10}\right) = 2.99$$

If there is not an asset market in the Radner equilibrium we have

$$\frac{u_{x_1}^1(s)}{u_{x_2}^1(s)} = \frac{u_{x_1}^2(s)}{u_{x_2}^2(s)} = p^{-1}(s), \text{ for } s = 1,2$$

but do not have

$$\frac{\pi(1)\, u_{x_h}^1(1)}{\pi(2)\, u_{x_h}^1(2)} = \frac{\pi(1)\, u_{x_h}^2(1)}{\pi(2)\, u_{x_h}^2(2)} = \frac{\lambda_h(1)}{\lambda_h(2)}, \text{for } h = 1,2$$

Conclusion: only have equal MRS across goods for the households. Do not have equal MRS across states. The equilibrium is not Pareto optimal.

b) Suppose there are complete markets, i.e. can buy at date 0 all goods of date 1 for all states. What is the equilibrium in this case?

Answer:

An equilibrium is a contingent price vector $(P_h(s))$ and an allocation $(x_h^i(s))$ such that:

1. $x_h^i(s) \in B_i(P(s))$ and there is no other $x_h'^i(s) \in B_i(P_h(s))$ that gives more utility to consumer i than $x^i(s)$. Where $B_i(P(s)) = \left\{(x^i) : \sum_h \sum_s P_h(s)\, x_h^i(s) \leq \sum_h \sum_s P_h(s)\, e^i(s)\right\}$.
2. Markets clear.

$$x_h^1(s) + x_h^2(s) = e_h^1(s) + e_h^2(s), \text{ for } s = 1,2$$

Household 1 solves the problem

$$\frac{1}{2}\left(4\ln x_1^1(1) + \ln x_2^1(1)\right) + \frac{1}{2}\left(4\ln x_1^1(2) + \ln x_2^1(2)\right)$$

s.t.

$$\sum_h \sum_s P_h(s)\, x_h^i(s) \leq P_h(s)\, e^i(s)$$

Observation: these are contingent prices which are different from the spot prices.

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$$\frac{4x_2^1(1)}{x_1^1(1)} = \frac{P_1(1)}{P_2(1)}$$

$$\frac{4x_2^1(2)}{x_1^1(1)} = \frac{P_1(1)}{P_2(2)}$$

$$\frac{4x_1^1(2)}{4x_1^1(1)} = \frac{P_1(1)}{P_1(2)}$$

and

$$\sum_h \sum_s P_h(s)\, x_h^1(s) \le \sum_h \sum_s P_h(s)\, e_h^1(s)$$

similar for consumer 2. Have 4 unknowns and 4 equations for household 1. Similarly have 4 unknowns and 4 equations for household 2. Clearing of markets

$$\begin{aligned} x_1^1(1) + x_1^2(1) &= \frac{4}{10}\left(1 + P_2(1) + 2P_2(2) + 2P_1(2)\right) + \\ &\quad \frac{1}{10}\left(2 + 2P_2(1) + P_2(2) + P_1(2)\right) \\ &= 3 \end{aligned}$$

$$\begin{aligned} x_1^1(2) + x_1^2(2) &= \frac{4}{10P_1(2)}\left(1 + P_2(1) + 2P_2(2) + 2P_1(2)\right) + \\ &\quad \frac{1}{10P_1(2)}\left(2 + 2P_2(1) + P_2(2) + P_1(2)\right) \\ &= 3 \end{aligned}$$

$$\begin{aligned} x_1^1(1) + x_1^2(1) &= \frac{1}{10P_2(2)}\left(1 + P_2(1) + 2P_2(2) + 2P_1(2)\right) + \\ &\quad \frac{4}{10P_2(2)}\left(2 + 2P_2(1) + P_2(2) + P_1(2)\right) \\ &= 3 \end{aligned}$$

where we have used the normalization $P_1(1) = 1$, (by Walras law).

The solution is $\{\frac{24}{10}, \frac{6}{10}, \frac{6}{10}, \frac{24}{10}, \frac{24}{10}, \frac{6}{10}, \frac{6}{10}, \frac{24}{10}\}$. $(P_1(1), P_1(2), P_2(1), P_2(2)) = (1,1,1,1)$.

c) Suppose there are only only spot commodity markets and Arrow-Debreu securities. What is the equilibrium in this case?

Answer:

Assume the security s pays 1 monetary unit in state s, zero in the other state. An equilibrium is a price vector $(P(s), q)$ and an allocation $(y^i, x^i(s))$ such that:

1. $(y^i(s), x^i(s)) \in B_i(P(s), q)$ and there is no other $(y'^i(s), x'^i(s)) \in B_i(P(s), q)$ that gives more utility to consumer i than $(y^i(s), x^i(s))$. Where $B_i(P(s), q) = \left\{(y^i, x^i) : \sum_s q_s y_s^i \le \sum_s q_s \left(\sum_h P_h(s)\, e_h^i(s)\right) \text{ and } \sum_h P_h(s)\, x_h^i(s) \le y_h^i\right\}$. These variables are exogenous.

2. Markets clear.

$$x^1(s) + x^2(s) = e^1(s) + e^2(s), \text{ for } s = 1, 2$$

$x^i(s)$ is the vector of goods consumed by household i at date 1 when the state is s.

In this case the equilibrium is $\{x_1^1(1) = \frac{24}{10}, x_2^1(1) = \frac{6}{10}, x_1^2(1) = \frac{6}{10}, x_2^2(1) = \frac{24}{10}, x_1^1(2) = \frac{24}{10}, x_2^1(2) = \frac{6}{10}, x_1^2(2) = \frac{6}{10}, x_2^2(2) = \frac{24}{10}\}$, $\{q(1) = q(2) = \frac{1}{2}\}$, $\{P_1(1) = P_2(1) = P_1(2) = P_2(2) = 2\}$.