Selecting Optimal Portfolios

1.

Expected Utility Theory (EUT)

1. Expected Utility Theory

- Foundations of Utility Theory
- Utility Functions and Their Properties
- Risk Tolerance Function and the Optimal Portfolio

1.1 Foundations of Utility Theory

- Learning objectives
- St. Petersburg Paradox
- Defining Utility
- Expected Utility Theory (EUT)
- Questions

Learning objectives

- say why mean-variance analysis is not sufficient,
- discuss the St Petersburg paradox,
- state the four axioms of a rational investor,
- state the rational expectations theorem,
- show that an investor deciding according to expected utility satisfies the four axioms.
- define a utility function,
- explain how utility functions are used to choose between investments.

The need for something more

- Even if we accept that investors only care about mean and variance, mean-variance analysis does not tell us which portfolio to hold.
- It only reduces the set of investments worth considering – from the full investment opportunity set it worth considering only the efficient portfolios.
- Within that set, it says nothing.
  Since that set –the efficient frontier – is generally a combination of line(s) and/or curve.
- We still do not have enough information to decide our investments.

We therefore need an extra concept
– on investor preferences – to go further.

The St Petersburg Paradox

How much is the right to play the following game worth?

- You keep on tossing a coin until it comes up tails.
- If there are n throws you receive 2^n roubles.
- The probability of terminating after exactly n throws is 2^{-n}.
- The expected pay-off is therefore

$$\sum_{n=1}^{\infty} 2^{-n} 2^{n} = \infty.$$

- If all one cares about is expectation then one should be willing to pay an arbitrarily large amount to play this game.
- This paradox goes back to at least the 18th century.

Interpreting the St Petersburg Paradox

- Practical experiments suggest that people are not willing to pay very much to play this game.
  $\Rightarrow$ In fact, 1.5 roubles is a typical response!

How can we explain people's reluctance to pay very much?

- One explanation is that not much value is ascribed to a very small probability of winning a very large amount of money.
- Another, related, explanation is that the prospect of getting two million dollars is not viewed as being twice as good as getting one million dollars.

Defining utility

- Interpreting utilities:
  - If $U(A) > U(B)$, A is preferable over B: $A \succ B$
  - If $U(A) = U(B)$, there is indifference between A and B: $A \sim B$
  - If $U(A) < U(B)$, B is preferable over A: $A \prec B$
- A utility function is, thus, a qualitative function.
- When it comes to investment choice applications of Utility Theory, it turns out that it is enough that utility functions map positive real numbers, representing total wealth at the end of the period W, to the real numbers.

$$U(W) : \mathbb{R}^+ \to \mathbb{R}$$

OBS: Utility is always defined in terms of investor's wealth W and not in terms of returns R.

Expected Utility Theory (EUT)

- We also need to be able to extend the standard Utility Theory under certainty, to the uncertain setup, as outcomes of investments are uncertain.
- This extension is due to Von-Newman and Morgernstern and is known as Expected Utility Theory (EUT).
- The key idea is that we should use the principle of maximising expected utility in investment decisions:
  - one chooses investing in the portfolio X over the portfolio Y, i.e. $X \succ Y$, if
$$\mathbb{E}(U(W_X)) > \mathbb{E}(U(W_Y)),$$
  - one is indifferent between portfolios X and Y, i.e. $X \sim Y$ if
$$\mathbb{E}(U(W_X)) = \mathbb{E}(U(W_Y)) ,$$

  where W_X refers to our total wealth if we adopt a certain investment strategy X, and W_Y again refers to a total wealth under a different strategy Y.

Modelling Investment Decisions

- Expected Utility Theory (EUT) is a convenient way to model investors' choices.
- However, it is not the only way.
- There are multiple ways to assess a model:
  - Does it correctly predict an investor's choices?
  - Are an investor's choices compatible with utility theory?
  - Does it follow from reasonable assumptions?

The rational investor

So why is EUT so popular?

Under some fairly mild assumptions – on the rationality of investors – one can prove that they make their decisions according to Expected Utility Theory (EUT).

A *rational investor* is one whose preferences satisfy the four axioms. These are:

1. Comparability
2. Transitivity
3. Independence
4. Certainty equivalence

Comparability

1. The first property is comparability

Given two investments, precisely one of

- $A \prec B$,
- $A \sim B$,
- $A \succ B$,

should hold.

OBS: This effectively states that the investor should always be able to express an opinion about the relative merits of two instruments.

Transitivity

2. Our second property is transitivity.

If A is preferred to B and B is preferred to C then A must be preferred to C. We also require that if $A \sim B$ and $B \sim C$ then $A \sim C$.

That is

$$\begin{aligned} A \succ B,\ B \succ C, &\implies A \succ C\ , \\ A \prec B,\ B \prec C, &\implies A \prec C\ , \\ A \sim B,\ B \sim C, &\implies A \sim C \end{aligned}$$

Independence

3. Another important property is independence.

If an investor is indifferent between A and B, and suppose we have a third investment C.

Let D be A with probability p, and C otherwise,

Let E be B with probability p, and C otherwise.

Independence states that in this case, the investor should be indifferent between D and E.

The idea is either that the investor receives C in both cases which clearly suggests indifference, or the investor receives one of two investments between which he is indifferent so again he should be indifferent.

Certainty equivalence

4. Another property sometimes used is certainty equivalence.

This states that the investor is indifferent between any investment and some guaranteed cash sum – the investment certainty equivalent.

Roughly stated, this says that every investment has an indifference price.

Certainty equivalence can be deduced from the other three axioms and the Archimedean axiom.

=> The Archimedean axiom roughly states that no investment is infinitely better than another investment.

Rational expectations theorem

Rational Expectations Theorem

An investor's preferences are given by expected utility if and only if their preferences satisfy the axioms of comparability, transitivity, independence and certainty equivalence.

- That EUT implies the four axioms is quite easy.
- That the four axioms imply expected utility is quite hard

Next we just check that EUT $\Longrightarrow$ each of the axioms.

EUT and Comparability

1. EUT $\Longrightarrow$ comparability

If preferences are given by expected utility then we simply take the investment with higher expected utility.

Since precisely one of

$$\mathbb{E}(U(A)) < \mathbb{E}(U(B)),$$
$$\mathbb{E}(U(A)) = \mathbb{E}(U(B)),$$
$$\mathbb{E}(U(A)) > \mathbb{E}(U(B)),$$

is true, we also have that precisely one of

$$A \prec B,\ A \sim B\ A \succ B,$$

is true, and comparability follows.

EUT and Transitivity

2. EUT $\Longrightarrow$ transitivity

If preferences are given by expected utility and $A \prec B \prec C$, then

$$\mathbb{E}((U(A)) < \mathbb{E}(U(B)) \text{ and } \mathbb{E}(U(B)) < \mathbb{E}(U(C)),$$

so

$$\mathbb{E}(U(A)) < \mathbb{E}(U(C))$$

and

$$A \prec C.$$

EUT and Independence

3. EUT $\Longrightarrow$ independence

The investments A and B are equivalent so

$$\mathbb{E}(U(A)) = \mathbb{E}(U(B)).$$

D is A with probability p and C with probability $1-p$

E is B with probability p and C with probability $1-p$

So

$$\begin{aligned}\mathbb{E}(U(D)) =& p\mathbb{E}(U(A)) + (1-p)\mathbb{E}(U(C)),\\ =& p\mathbb{E}(U(B)) + (1-p)\mathbb{E}(U(C)),\\ =& \mathbb{E}(U(E)).\end{aligned}$$

EUT and Certainty equivalence

4. EUT $\Longrightarrow$ certainty equivalence

For this one we need the utility function, U, to be increasing and continuous. Given these properties, the function U has an inverse U^{-1}.

For an investment A, we set

$$C = U^{-1}(\mathbb{E}(U(A)).$$

Note C is a constant, so it bears no risk.

We then have

$$\mathbb{E}(U(C)) = U(C) = \mathbb{E}(U(A)),$$

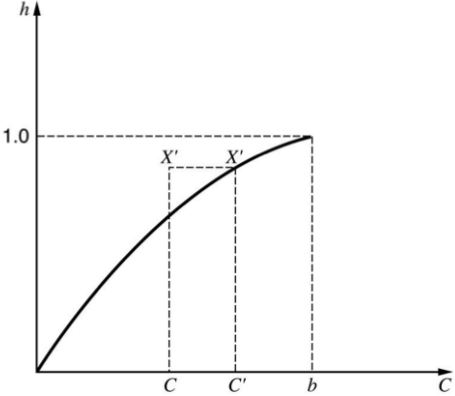
so the investor is indifferent between C and A, as required.

Certainty equivalence

We can also see the certainty equivalence in the following way:

GAME= (b with prob. h
0 with prob. 1-h)

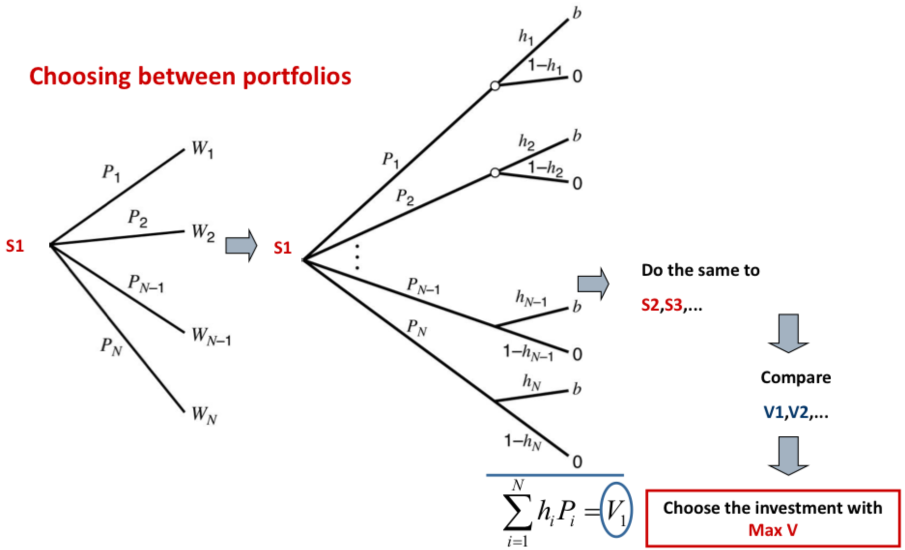
C= **certain equivalent**



For any fixed C, we can:

- Fix h and evaluate b that guarantees indifference, or
- Fix b and evaluate h that guarantees indifference.

Certainty equivalence



Example: EUT decisions

$$\underset{k}{Max}\, E[U(W_i)]_k = \sum_{i=1}^{n} P_i(W_i)U(W_i)$$

$$U(W_i) = 4W_i - \left(\frac{1}{10}\right)W_i^2$$

with
i – counter for wealth alternatives
n – totam number of wealth alternatives
E[U(Wi)]k – final wealth expected utility associated to project *k*;
U(Wi) – utility index associated to each final wealth level
(which is subjacent to the return of project *k*);
Pi(Wi) – probability associated to each wealth level in project *k*.

A		B		C	
Prob.	Return	Prob.	Return	Prob.	Return
3/15	20	1/5	19	1/4	18
5/15	18	2/5	10	1/4	16
4/15	14	2/5	5	1/4	12
2/15	10			1/4	8
1/15	6				

	U(Wi)
5	17,5
6	20,4
8	25,6
10	30,0
12	33,6
14	36,4
16	38,4
18	39,6
19	39,9
20	40,0

$E[U(W)]_A = 36{,}3$ $E[U(W)]_B = 26{,}98$ $E[U(W)]_C = 34{,}4$

Theory questions

1. What four axioms does a rational investor's behaviour satisfy?
2. What does the rational expectations theorem say?
3. What does the axiom of comparability say? Show that an investor deciding according to expected utility satisfies this axiom.
4. What does the axiom of transitivity say? Show that an investor deciding according to expected utility satisfies this axiom.
5. What does the axiom of independence say? Show that an investor deciding according to expected utility satisfies this axiom.

1.2 Utility Functions and their Properties

- Learning Objectives
- Properties of Utility Functions
- Indifference pricing
- Risk aversion and curvatures: measuring absolute and relative risk aversion
- Questions

Learning objectives

- relate the properties of utility functions to investor behaviour,
- state when two utility functions are equivalent,
- give some examples of utility functions.
- define indifference prices and risk premia,
- compute indifference prices.
- define and derive absolute risk aversion,
- define and derive relative risk aversion,
- classify the risk profile of investors given his/her utility function.

Properties of Utility Functions

- We cannot observe utility functions!
- Facing a particular investor we need to choose and utility function that fits his preferences .
- Utility are just qualitative functions.
- Utility functions are only needed as a tool to decide the optimal (for a particular investor) investment strategy.
- We only care about the ranking of alternative investments.
- Two utility functions are said to be equivalent if they lead to the same decisions, and it that case any such function would do the job.

OBS: To be able to assign an mathematical function $U(\cdot)$ to model the preferences of a particular investor, we need to what how to interpret the mathematical properties of $U(\cdot)$ in terms of risk profiles.

First derivative: $U'(\cdot)$

- Investors will generally prefer more to less.
- So we require,

$$W_X < W_Y \implies U(W_X) < U(W_Y),$$

i.e., U is increasing $\implies U'(W) = \frac{\partial U}{\partial W} > 0.$

OBS: A decreasing $U(\cdot)$ would therefore say that the investor actually prefer less money under certain circumstances.

Second derivative: $U''(\cdot)$

- We also need to understand what different U functions may mean in terms of the investor's atitude towards risk.
- Let us consider two investments X and Ysuch that, W_X is risky, but W_Y is not, and

$$\mathbb{E}(W_X) = \mathbb{E}(W_Y) = W_Y$$

  - A risk-neutral investor would not care about varianceso, the investor would be indifferent between the two investments, $X \sim Y$, and

$$\mathbb{E}(U(W_X)) = \mathbb{E}(U(W_Y)).$$

  - However, the risk averse investor would prefer Y to X, i.e. $X \succ Y$ and

$$\mathbb{E}(U(X)) < \mathbb{E}(U(Y)).$$

  - Finally, the risk lover investor would instead prefer X to Y, i.e. $X \prec Y$ and

$$\mathbb{E}(U(X)) > \mathbb{E}(U(Y)).$$

Second derivative: $U''(\cdot)$

What property on $U(\cdot)$ different attitudes towards risk imply?

- Suppose we have $W_A < W_Y < W_B$, and p is such that

$$W_Y = (1-p)W_A + pW_B.$$

- Let X pay W_A with probability $1-p$ and W_B with probability p. We then have

$$\mathbb{E}(W_X) = \mathbb{E}(W_Y) = W_Y.$$

- But X is risky whereas Y is not, so recall
  - A risk-neutral investor would be indifferent Y over X: $X \sim Y$
  - A risk-averse investor would therefore choose Y over X : $X \prec Y$
  - A risk-loving investor would therefore choose X over Y: $X \succ Y$

Utility curvature

- Let us take the case of the risk averse, we want

$$\mathbb{E}(U(W_X)) < \mathbb{E}(U(W_Y)) = U(W_Y) \; .$$

- This is equivalent to

$$(1-p)U(W_A) + pU(W_B) < U(W_Y) \; .$$

- However, this is precisely the definition of (strict) concavity, since it states that points on the graph of U between W_A and W_B will lie above the chord from A to B.
- Thus, a risk averse investor will have a concave utility function.

$$U''(W) = \frac{\partial^2 U}{\partial W^2} < 0$$

Utility curvature

- If the utility function was a straight line then we would have

$$U''(W) = \frac{\partial^2 U}{\partial W^2} = 0 \ ,$$

$$\mathbb{E}(U(W_X)) = \mathbb{E}(U(W_Y)) \ ,$$

and the investor is then said to be risk-neutral.

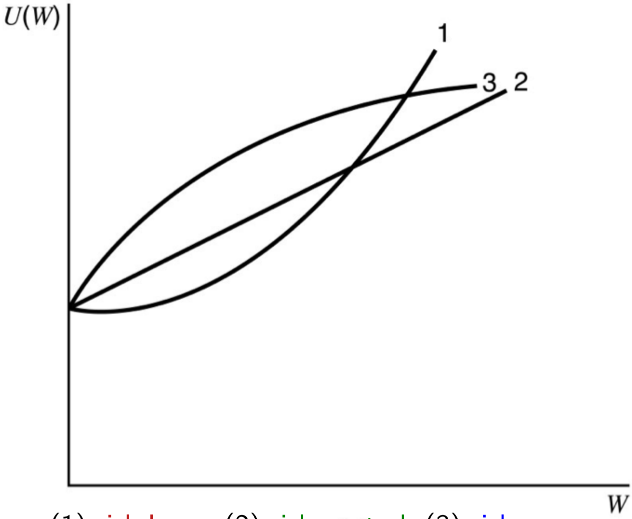
- If the utility function is convex then we have

$$U''(W) = \frac{\partial^2 U}{\partial W^2} > 0 \ ,$$

$$\mathbb{E}(U(W_X)) > \mathbb{E}(U(W_Y)) \ ,$$

and the investor prefers a risky asset with the same expectation to a non-risky one, and is said to be risk-seeking.

Utility curvature



(1) risk lover, (2) risk neutral, (3) risk averse

Examples of utility functions

Some typical utility functions are

- $U(W) = \log(W)$, log utility
- $U(W) = 1 - e^{-W}$, exponential utility
- $U(W) = aW - bW^2$, with $b > 0, W \leq \frac{a}{2b}$, quadratic utility

OBS: All the above functions are concave, i.e. only appropriate for risk averse investors. HW: Suggest good utility functions for risk lovers.

Example: log utility and the St Petersburg paradox

- Recall St Peterburg paradox Suppose we take a log utility function, the utility then ascribed to a value W is $\log(W)$.
- So the expected utility is

$$\mathbb{E}(\log V) = \sum_{n=1}^{\infty} \log(2^n) 2^{-n}, = \sum_{n=1}^{\infty} n \log(2) 2^{-n}.$$

OBS: This is finite and not too hard to compute (exercise for the enthusiastic...)

Equivalence

Theorem

If U is a utility function and we take

$$V(W) = a + bU(W)$$

with $a, b \in \mathbb{R}$, and $b > 0$, then U and V are equivalent.

Proof. If

$$\mathbb{E}(U(W_X)) > \mathbb{E}(U(W_Y))$$

then

$$a + b\mathbb{E}(U(W_X)) > a + b\mathbb{E}(U(W_Y)),$$

so

$$\mathbb{E}(V(W_X)) > \mathbb{E}(V(W_Y)).$$

U and V lead to the same investment decisions, they are equivalent. ∎

Indifference pricing

- For an initial wealth W_0, we can think of investments as a choice between:
  - investing in a portfolio which changes our wealth by a random variable X, or
  - putting it into something worth a fixed amount C
- The value of C which makes

$$\mathbb{E}(U(W_0 + X)) = \mathbb{E}(U(C)) = U(C),$$

  is the wealth at which the investor is indifferent.
- The value of X to the investor is then $IP(X) = C - W_0$ and is called the indifference price for X. This could be either positive or negative.
- Since is always U increasing it will be invertible, so we can write

$$C = U^{-1}(\mathbb{E}(U(W_0 + X))).$$

Risk premia

- If the utility function is linear then

$$U(W) = aW + b, \quad a > 0,$$

then

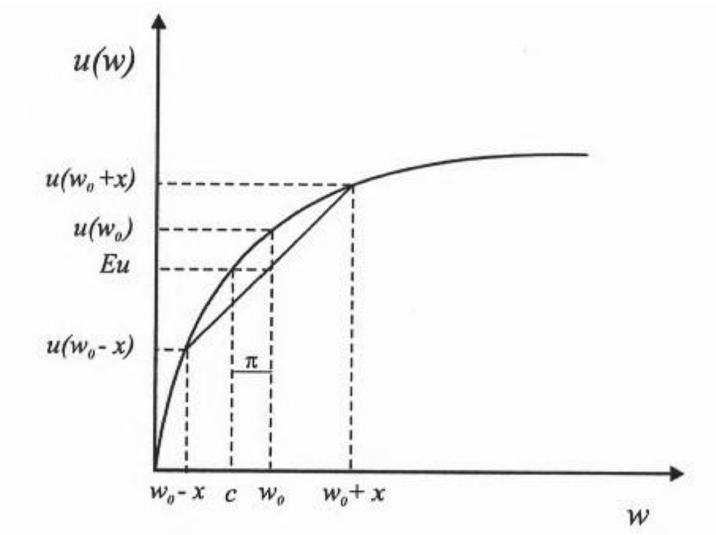
$$\mathbb{E}(U(W_0 + X)) = a(W_0 + \mathbb{E}(X)) + b = U(W_0 + \mathbb{E}(X)) \ .$$

So the indifference price is $IP(X) = \mathbb{E}(X)$.

- For a general utility function, we define the risk premium to be the difference between what a risk-neutral investor would pay and the non-neutral indifference price so it equals

$$\pi = \mathbb{E}(X) - (C - W_0) = \mathbb{E}(W) - C$$

Illustration



Indifference price

Example:

- Suppose an investor has 100 000 and has a log utility function.
- Consider an investment, Y, that pays 150, or -50 with probability 0.5.

What is the indifference price?

We need to know

- initial wealth,
- utility function,
- distribution of final value of investment.

Computing the indifference price

Example (cont.): We need to:

- Compute $\mathbb{E}(\log(100\ 000 + Y))$, since we have

$$\log(100\ 000 + 150) = 11.51442434\ ,$$

$$\log(100\ 000 - 50) = 11.51242534\ ,$$

  the expected utility is $\mathbb{E}(U(W)) = 11.51342484$.
- and then exponentiate – since exp is the inverse of log – to get the indifference wealth is $C = 100\ 049.95$.
- This means that the indifference price of Y is

$$IP(Y) = C - W_0 = 49.95.$$

- And the risk premium is $\pi = \mathbb{E}(Y) - IP(Y) = 0.05$.

Modified example

- Suppose instead the wealth was 1 000.
- We now must compute $\mathbb{E}(\log(1\ 000 + Y))$

$$\log(1\ 000 + 150) = 7.047517221,$$

$$\log(1\ 000 - 50) = 6.856461985\ ,$$

to get the expected utility $\mathbb{E}(U(W)) = 6.951989603$.

- The indifference wealth is $C = 1\ 045.227248$.
- This means that the indifference price is $C - W_0 = 45.23$.
- The risk premium has increased to $\pi = 4.77$.

Equivalence and curvature

- We know that concavity leads to risk aversion.
- We also know that replacing the function $U(W)$ by $aU(W) + b$ leads to identical decisions and so identical indifference prices.

Searching for a measure of risk aversion:

- We would expect that the making U'' more negative would increase risk premia.
- But, since U and $V = aU + b$ give the same preferences, any attempt to quantify risk aversion must assign the same risk aversion to both these functions.

$$\Downarrow$$

Absolute Risk Aversion

- If we consider $V = aU + b$
- Differentiating makes the b disappear: $V' = aU'$
- Differentiating once more we get: $V'' = aU''$
- To get rid of the a we can take ratios: $\frac{V''}{V'} = \frac{aU''}{aU'} = \frac{U''}{U}$
- Since $U'' < 0$ the fraction $-\frac{U''}{U}$ is positive and can be seen as a measure of risk aversion, and this is the same for U and V.

Absolute Risk Aversion (ARA)

$$A(W) = \frac{-U''(W)}{U'(W)}$$

OBS: It turns out the absolute risk aversion, at a given level of wealth, tells us how much to multiply the variance of an investment by to get the risk premium (check the proof in the textbook).

Interpreting ARA

- If $A'(W) < 0$, as wealth increases the lower it is the degree of ARA . *The higher the wealth the higher the amount (in euros) one is willing to invest in risky assets.*

- If $A'(W) = 0$ that is ARA is constant then the risk premium does not vary with wealth.
  No matter the wealth level one invests always the same amount (in euros) in risky assets.

- If $A'(W) > 0$ as wealth increases the higher it is the degree of ARA . *The higher the wealth the lower the amount (in euros) one is willing to invest in risky assets.*

Example: ARA for log utility

Suppose we take log-utility, i.e,

$$U(W) = \log W.$$

Then

$$U'(W) = \frac{1}{W},$$
$$U''(W) = -\frac{1}{W^2}.$$

We therefore have

$$A(W) = \frac{1}{W} \quad \Longrightarrow \quad A'(W) = -\frac{1}{W^2} < 0$$

so, we get a decreasing ARA function.

Example: ARA for exponential utility

Suppose we take exponential-utility, i.e,

$$U(W) = 1 - e^{-aW} \text{ with } a > 0.$$

Then

$$\begin{aligned} U'(W) =& ae^{-aW}, \\ U''(W) =& - a^2 e^{-aW}. \end{aligned}$$

We therefore have

$$A(W) = a \quad \Longrightarrow \quad A'(W) = 0$$

and we get a constant ARA.

ARA and utility functions

Condition	Definition	Property of $A(W)^a$	Example^b
Increasing absolute risk aversion	As wealth increases hold fewer dollars in risky assets	$A'(W) > 0$	W^{-CW^2}
Constant absolute risk aversion	As wealth increases hold same dollar amount in risky assets	$A'(W) = 0$	$-e^{-CW}$
Decreasing absolute risk aversion	As wealth increases hold more dollars in risky assets	$A'(W) < 0$	$\ln W$

[a] $A'(W)$ is the first derivative of $A(W)$ with respect to wealth.
[b] The proof is left to the reader.

Relative risk aversion

It is also useful to think in terms of relative risk aversion where the aversion is in terms of fractions or proportions of current wealth that might be lost instead of absolute amounts.

Relative Risk Aversion (RRA)

$$R(W) = -\frac{WU''(W)}{U'(W)}.$$

OBS: Note that an investor with constant absolute risk aversion will display increasing relative risk aversion.

Interpreting RRA

- If $R'(W) < 0$, as wealth increases the lower it is the degree of RRA . *The higher the wealth, the higher is the proportion (in %) one is willing to invest in risky assets.*

- If $R'(W) = 0$ that is RRA is constant, so the degree of RRA is the same no matter the level of wealth.
No matter the wealth, one invests always the same proportion (in %) in risky assets.

- If $R'(W) > 0$ as wealth increases the higher it is the degree of RRA . *The higher the wealth, the lower the proportion (in %) one is willing to invest in risky assets.*

Examples: RRA for log and exponential utility

- We saw that for log utility, $A(W) = W^{-1}$, so the associated relative risk aversion is

$$R(W) = WA(W) = 1 \quad \Longrightarrow \quad R'(W) = 0 \; ,$$

  so, we have a constant RRA.

- On the other hand, for exponential utility, the relative risk aversion is equal to

$$R(W) = WA(W) = aW \quad \Longrightarrow \quad R'(W) = a > 0$$

  so, in this case RRA increases with wealth levels.

RRA and utility functions

Condition	Definition	Property of $R'(W)$	Examples of Utility Functions
Increasing relative risk aversion	Percentage invested in risky assets declines as wealth increases	$R'(W) > 0$	$W - bW^2$
Constant relative risk aversion	Percentage invested in risky assets is unchanged as wealth increases	$R'(W) = 0$	$\ln W$
Decreasing relative risk aversion	Percentage invested in risky assets increases as wealth increases	$R'(W) < 0$	$-e^{2W-1\backslash 2}$

Theory questions

1. What properties would you expect a utility function to have and why?
2. What does it mean for two utility functions to be equivalent?
3. Give examples for three typical utility functions.
4. If an investor is risk-neutral, what can we say about his utility function?
5. If an investor is risk-averse, what can we say about his utility function?
6. Define the indifference price.
7. Define the risk premium of an investment.

Theory questions

9. Define and derive the absolute risk aversion function associated to a utility function.
10. Define and derive the relative risk aversion function associated to a utility function.
11. How do we compute the indifference price given the absolute risk aversion?
12. How do we compute the indifference price given the relative risk aversion?
13. Suppose an investor has constant absolute risk aversion, what does this tell us about this behaviour?

1.3 Risk Tolerance Function and the Optimal Portfolio

- Learning Objectives
- Risk Tolerance Functions
- Finding optimal portfolios
- Quadratic utility and portfolio theory
- Questions

Learning objectives

- establish the connection between utility functions and risk tolerance functions (RTFs).
- understand the difficulties associated with deriving closed-form RTFs.
- relate quadratic RTFs to mean-variance analysis,
- find second-order Taylor approximations to utility functions and the associated quadratic RTFs.
- for closed-form RTFs directly determine optimal portfolios.
- use indifference curves of RTFs to find optimal portfolios.

Maximal Expected Utility Principle

- Let us use EUT in MVT context.
- MVT allows us to get the set of efficient portfolios one should consider.
- EUT tells us we should use the maximal expected utility principle, to find the optimal portfolio – the one the investor prefers over all others.

Formally we have

$$\max_{p} \quad \mathbb{E}\left[U(W)\right]$$
$$\text{s.t.} \quad p \in EF$$

Getting it all together

- MVT efficient frontiers are defined in $(\sigma, \bar{R})$.
- It would be nice to redefine $\mathbb{E}[U(W)]$ as a function of $(\sigma, \bar{R})$.

Risk Tolerance function (RTF)

The RTF $f : (\sigma, \bar{R}) \to \mathbb{R}$ is defined as

$$f(\sigma, \bar{R}) = E(U(W)).$$

RTF indifference curves are the level curves for which

$$f(\sigma, \bar{R}) = K$$

for some fixed expected utility level K.

OBS: The above definition does not guarantee that RTF are easy to obtain in closed-form.

RTF: quadratic utility

Sometimes we can do it ...

$$
\begin{aligned}
f(\sigma, \bar{R}) &= E(U(W)) \\
&= \mathbb{E}(W - bW^2), \\
&= \mathbb{E}(W_0(1+R)) - b\mathbb{E}(W_0^2(1+R)^2), \\
&= W_0(1+\mathbb{E}(R)) - bW_0^2\mathbb{E}(1+2R+R^2), \\
&= W_0(1+\bar{R}) - bW_0^2\left(1+2\bar{R}+\mathbb{E}(R^2)\right) \\
&= W_0(1+\bar{R}) - bW_0^2\left(1+2\bar{R}+\sigma^2+\bar{R}^2)\right) \\
&= -bW_0^2(\sigma^2+\bar{R}^2) + W_0(1-2bW_0)\bar{R} + W_0(1-bW_0)
\end{aligned}
$$

- where we have used $W = W_0(1+R)$, and
- the statistical property $\sigma^2 = \mathbb{E}(R^2) - \bar{R}^2$.

OBS: This means that for quadratic investors, choice between portfolios is purely determined by expected return and volatility.

RTF: log utility

Sometimes we get stuckt ...

$$
\begin{aligned}
f(\sigma, \bar{R}) &= E(U(W)) \\
&= \mathbb{E}(\log(W)) \\
&= \mathbb{E}(\log(W_0(1+R))) \\
&= \log(W_0) + \underbrace{\mathbb{E}(\log(1+R))}_{\text{this cannot be written in term of } \sigma, \bar{R}.}
\end{aligned}
$$

What can we do when this happens?

1. Add the assumptions that returns follow a distribution for which $\sigma, \bar{R}$ are sufficient statistics.
2. Numerically evaluate it.
3. Approximate it.

Approximating RTFs

- One justification for quadratic utility is that it can be viewed as an approximation to any other utility function.
- Two functions U and V agree to second order at W_0 if

$$U(W) - V(W) = o((W - W_0)^2,$$

where $o((W - W_0)^2)$ means something small compared to $(W - W_0)^2$, i.e.

$$\frac{U(W) - V(W)}{(W - W_0)^2} \to 0$$

as $W \to W_0$.

Taylor and quadratic utility

- If U is a general utility function, we can always approximate by a quadratic:

$$U(W) = U(W_0) + U^{'}(W_0)(W - W_0) + U^{''}(W_0)(W - W_0)^2/2 + o((W - W_0)^2).$$

- And we can derive its second-order Taylor expansion around W_0

$$U(W) \approx U(W_0) + U^{'}(W_0)(W - W_0) + \frac{1}{2}U^{''}(W_0)(W - W_0)^2 \ .$$

- Note the above approximation is always quadratic, for any general utility U.
- As long as $W - W_0$ is small the approximation will be good.

Equivalence of RTFs

- Risk tolerance functions (RTFs) just like utility functions are qualitative functions.
- Two RTFs that lead to the same ranking of portfolios in the $(\sigma, \bar{R})$–space are considered to be equivalent as they lead to the same investment decisions. An important result is

Theorem

The RTF resulting from a second-order Taylor approximation of a generic utility function U is equivalent to

$$f(\bar{R}, \sigma) = \bar{R} - \frac{1}{2} r_0 \left[\bar{R}^2 + \sigma^2 \right] \ ,$$

where r_0 is the coefficient of relative risk aversion evaluated at W_0.

HW: Formally show this.

Graphically representing RTF

- Note RTF has domain in our usual space $(\sigma, \bar{R})$. To represent it graphically we would need to be able to do a 3D representation.
- Alternatively we can use the ideia of level curves.
- We can plot curves where all investments have the same level of expected utility => indifference curves
- For closed-form RTFs – expressed in terms of σ and $\bar{R}$ – we can turn the equation round to get:
  - σ as a function of $\bar{R}$ and a fixed expected utility level K,

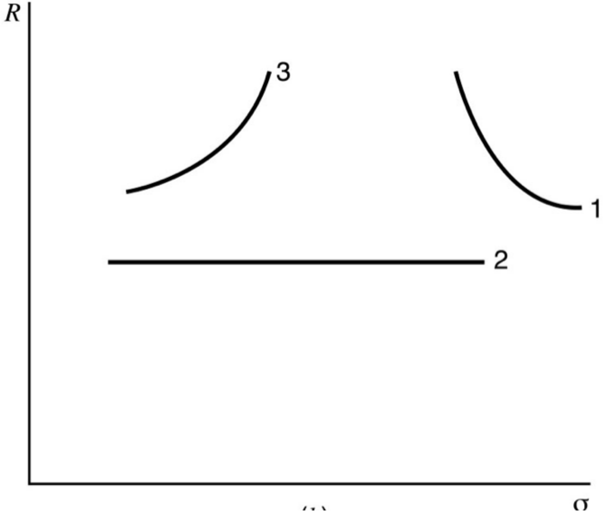
$$f(\sigma, \bar{R}) = K \quad \Longrightarrow \quad \sigma = IC(\bar{R}, K).$$

  - OR, $\bar{R}$ as a function of σ and a fixed expected utility level K

$$f(\sigma, \bar{R}) = K \quad \Longrightarrow \quad \bar{R} = IC(\sigma, K).$$

for fixed K – varying $\bar{R}$ or σ – we get indifference curves.

Indifference curves



(1) risk lovers; (2) risk lovers ; (3) risk averse

Finding Optimal Portfolios

We need to find the point on the efficient frontier that maximizes the RTF

$$\max_{p} \quad f(\sigma_p, \bar{R}_p)$$
$$\text{s.t.} \quad p \in EF$$

1. We can use direct maximisation of RTF
2. We can use indifference curves.

Finding Optimal Portfolios

1. Use direct maximisation of RTF

$$\max_{p} \quad f(\sigma_p, \bar{R}_p)$$
$$\text{s.t.} \quad p \in EF$$

- Recall the EF can be written as:

$$\sigma_p = EF(\bar{R}_p) \qquad \text{or} \qquad \bar{R}_p = EF(\sigma_p)$$

- So including the restriction, the problem reduces to:

$$\max_{\bar{R}_p} \quad f(EF(\bar{R}_p), \bar{R}_p) \qquad \text{or} \qquad \max_{\sigma_p} \quad f(\sigma_p, EF(\sigma_p))$$

Finding Optimal Portfolios

2. Using indifference curves

We need that the slope of our indifference curves (IC) and that of the efficient frontier (EF) match in the $(\sigma, \bar{R})$ space.

Since we have

$$\sigma_p = EF(\bar{R}_p) \qquad \text{or} \qquad \bar{R}_p = EF(\sigma_p)$$

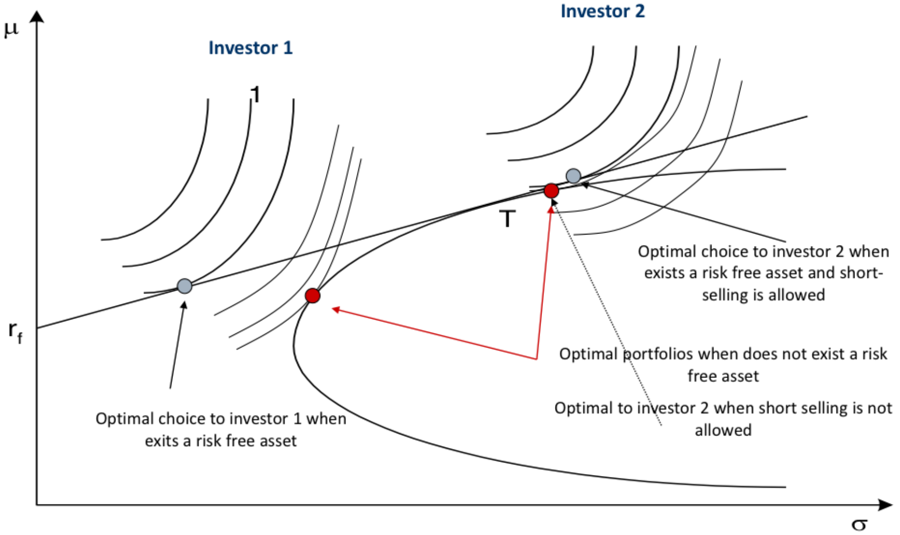
and

$$\sigma_p = IC(\bar{R}_p, K) \qquad \text{or} \qquad \bar{R}_p = IC(\sigma_p, K)$$

So, optimal portfolios solve

$$\frac{\partial EF}{\partial \bar{R}_p} = \frac{\partial IC}{\partial \bar{R}_p} \qquad \text{or} \qquad \frac{\partial EF}{\partial \sigma_p} = \frac{\partial IC}{\partial \sigma_p}$$

Optimal Portfolios using IC



Theory questions

1. Define risk tolerance functions(RTFs) in terms of utility functions.
2. Derive and interpret the indifference curves associated with a given RTF.
3. What can you conclude about the shape of indifference curves or risk averse, risk neutral and risk loving investors, in the $(\sigma, \bar{R})$ – space?
4. Why are quadratic RTFs so important in mean-variance analysis?
5. Given the equation(s) for the efficient frontier (EF) and a RTF, how to find the optimal investment?
6. Given the equation(s) for the efficient frontier (EF) and a set of indifference curves (IC), how to find the optimal investment?