$$\max_{x_A, x_B} \boxed{\frac{\bar{R}_p - R_f}{\sigma_p}}$$

Repetitorium

s.t. $\bar{R}_p = x_A \bar{R}_A + x_B \bar{R}_B$

- $\sigma_p = (x_A^2\sigma_A^2 + x_B^2\sigma_B^2 + 2x_Ax_B\sigma_{AB})^{1/2}$
- $x_A + x_B = 1$

insert constraints into take FOC derivative = 0

$$\begin{pmatrix} R_A - R_f \\ R_B - R_f \end{pmatrix} = \begin{pmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{pmatrix} \begin{pmatrix} z_A \\ z_B \end{pmatrix}$$

$\Leftrightarrow \quad R - R_f 1 = Vz \quad \Leftrightarrow \quad z = V^{-1}(E - R_f 1)$

$x_A = \frac{z_A}{\Sigma z_i} \quad x_B = \frac{z_B}{\Sigma z_i}$

Question: Risk free rate is 6%
Expected rate of market return is 12%
Security XYZ beta of 1.5
What is the expected return of XYZ according to CAPM (Capital Asset Pricing Model)?

Solution: CAPM: $\underbrace{E[r_{xyz}] - r_f}_{\text{Excess return XYZ}} = \beta\underbrace{(E[r_m] - r_f)}_{\text{Excess return the market port.}}$

$E[r_{xyz}] - 6\% + 1.5(12\% - 6\%) = \underline{15\%}$

SLIDE 10 –

Lemma 1

(i) $Cov(R_i, R_j) = \sum_{j=1}^{N} \overset{\sigma_{ij}}{V_{ij}} \, Cov(R_i, R_j)$

(ii) $\sum_{j=1}^{N} x_j \, Cov(R_i, R_j) = \underline{(VX)_i}$ where x is the tangency portfolio

Proof

(i) Recall $R_T = \sum_{j=1}^{N} x_j R_j \Rightarrow \bar{R}_T = \sum_{j=1}^{N} x_j \bar{R}_j$

$Cov(R_i, R_j) \underset{\text{definition}}{=} E[(R_i - \bar{R}_i)(R_j - \bar{R}_j)] \overset{\text{insert } R_T, \bar{R}_T}{=}$

$= E[(\sum_k x_k R_k - \sum_k x_k \bar{R}_k)(R_j - \bar{R}_j)] =$

$= E[\sum_k x_k (R_k - \bar{R}_k)(R_j - \bar{R}_j)] = \sum_k x_k \underbrace{E[(R_k - \bar{R}_k)(R_j - \bar{R}_j)]}_{Cov(R_k, R_j)}$

Proof (ii) $\left(\sum_j x_j \, Cov(R_i, R_j) = (VX)_i \right)$

for $n = 2$

$$VX = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 x_1 + \sigma_{12} x_2 \\ \sigma_{12} x_1 + \sigma_2^2 x_2 \end{pmatrix} =$$

$$= \begin{pmatrix} \sum_{j=1}^{2} x_j \, Cov(R_j, R_1) \\ \sum_{j=1}^{2} x_j \, Cov(R_j, R_2) \end{pmatrix}$$

use $Cov(R_i, R_i) = Var(R_i)$

SLIDE 10-1

Lemma 2

$$E[R_i] = R_f + \lambda \, Cov(R_i, R_T)$$

Proof

From IPM II 2_B slide 10 1 we know

$\bar{R} - R_f 1 = Vz$ where $z = \lambda x \quad \lambda \in \mathbb{R}$

$$\begin{pmatrix} \bar{R}_1 - R_f \\ \bar{R}_2 - R_f \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

So $\bar{R}_i - R_f = (Vz)_i$

$= E[R_i] = R_f + (Vz)_i = R_f + \lambda (VX)_i = R_f + \lambda \, Cov(R_i, R_T)$

Lemma 1

Lemma 3 (Finding λ)

$$\lambda = \frac{E[R_T] - R_f}{Var(R_T)}$$

Proof

$E[R_T] \underset{\text{def. } R_T}{=} E[\sum_{j=1}^{N} x_j R_j] = \sum_{j=1}^{N} x_j E[R_j] \overset{\text{Lemma 2}}{=}$

$= \sum_j x_j (R_f + \lambda \, Cov(R_j, R_T)) = \sum_j x_j R_f + \lambda \sum_j x_j Cov(R_j, R_T)$

$= R_f \sum_j x_j + \lambda \underbrace{Cov(R_T, R_T)}_{Var(R_T)} = R_f + \lambda \, Var(R_T)$

Solve for $\lambda = \frac{E[R_T] - R_f}{Var(R_T)}$

So from Lemma 2 and 3

Lemma 2: $E[R_i] = R_f + \lambda \, Cov(R_i, R_T)$

Lemma 3: $\lambda = \frac{E[R_T] - R_f}{Var(R_T)}$

$\Rightarrow E[R_i] = R_f + \frac{Cov(R_i, R_T)}{Var(R_T)} (E[R_T] - R_f)$

if we run put $T = M$ then

$E[R_i] - R_f = \underbrace{\frac{Cov(R_i, R_M)}{Var(R_M)}}_{= \beta_{i,M}} (E[R_M] - R_f)$

This is CAPM

From CAPM: $E[r_i] - r_f = \beta(E[r_m] - r_f)$

for $i = m$

$E[r_m] - r_f = \beta E[r_m] - r_f$

$\Leftrightarrow$

$\beta = 1$

CAPM:

$E[r] = r_f + \beta(E[r_m] - r_f)$

$-5 + 1.2(\frac{17}{10} - 5) =$

$= 5 + 12 - 17$

$E[r] - 20 =$

$= CAPM + 3\%$

$= CAPM + \alpha$

$\alpha > 0$

good buy!

CAPM

$E[r_i] - r_f = \beta(E[r_m] - r_f)$

$\bar{R}_i = \beta \bar{R}_m$

Linear regression model

$\bar{R}_i = \alpha + \beta \bar{R}_M$

apple α_{CAPM} section

$R_i = \alpha + \beta R + \varepsilon$

constant term / error term

compare to single index model

CAPM $\alpha = 0$ equilibrium

Standard CAPM

$E[r_i] = r_f + \beta(E[r_m] - r_f)$

factor