Lisbon University
Lisbon School of Economics and Management
Ms in Economics, Mathematical Finance and
Monetary and Financial Economics

Mathematical Economics – 1st Semester - 2023/2024

Exercises - Group IV

1. Solve the following variational problems:
   (a) $\max \int_0^1 (4xt - \dot{x}^2)\, dt,\ x(0) = 2,\ x(1) = 2/3$
   (b) $\min \int_0^1 (t\dot{x} + \dot{x}^2)\, dt,\ x(0) = 1,\ x(1) = 0$
   (c) $\min \int_0^1 (x^2 + 2tx\dot{x} + \dot{x}^2)\, dt,\ x(0) = 1,\ x(1) = 2.$

2. Consider the planar curve $\gamma(x) = (x, f(x))$ with $f \in C^1$ that connects the points $A = (x_0, y_0)$ and $B = (x_1, y_1)$. The length of the curve γ is given by

$$L(\gamma) = \int_{x_0}^{x_1} \sqrt{1 + (f'(x))^2}\, dx$$

   Determine the curve γ which minimizes the length between A and B. Formulate the problem as a variational problem and solve it using the calculus of variations.

3. Solve the variational problems:
   (a)
$$\min \int_0^1 (t\dot{x} + \dot{x}^2)\, dt, \quad x(0) = 1, \quad x(1) \geq 1$$
   (b)
$$\max \int_0^1 (10 - \dot{x}^2 - 2x\dot{x} - 5x^2)e^{-t}\, dt \quad x(0) = 1, \quad x(1) \text{ free}$$

4. Let $A(t)$ denote a pensioner's wealth at time t and w be the pension income (constant) per unit time. The pensioner consumption is given by

$$C(t) = rA(t) + w - \dot{A}(t),$$

   where $0 < r < 1$. Now the pensioner wants to maximize

$$\int_0^T U(C(t))e^{-rt}\, dt$$

   knowing that $A(0) = A_0$. The pensioner's utility function U is given by $U(C) = 1 - e^{-C}$. Determine the optimal consumption $C(t)$ so that at the end of the period the pensioner retains at least $2A_0$, i.e., $A(T) \geq 2A_0$.

5. Solve the following optimal control problems

(a)

$$\max_{u(t)\in\mathbb{R}} \int_0^2 (e^t x(t) - u(t)^2)\,dt, \quad \dot{x} = -u(t), \quad x(0) = 0, \quad x(2) \text{ free}$$

(b)

$$\max_{u(t)\in\mathbb{R}} \int_0^1 (1 - u(t)^2)\,dt, \quad \dot{x} = x(t) + u(t), \quad x(0) = 1, \quad x(1) \text{ free}$$

(c)

$$\min_{u(t)\in\mathbb{R}} \int_0^1 (x(t) + u(t)^2)\,dt, \quad \dot{x} = -u(t), \quad x(0) = 0, \quad x(1) \text{ free}$$

6. Solve the following optimal control problems

(a)

$$\max_{u(t)\in\mathbb{R}} \left\{ \int_0^1 -\frac{1}{2}u(t)^2\,dt + \sqrt{x(1)} \right\}, \quad \dot{x} = x + u, \quad x(0) = 0, \quad x(1) \text{ free}$$

(b)

$$\max_{u(t)\in\mathbb{R}} \left\{ \int_0^T -e^{-t}(x(t) - u(t))^2\,dt - e^{-T}x(T)^2 \right\}, \quad \dot{x} = u - x + 1, \quad x(0) = 0, \quad x(T) \text{ free}$$

7. Solve the following optimal control problems

(a)

$$\max_{u(t)\in\mathbb{R}} \left\{ \int_0^{+\infty} 2\sqrt{x(t) - u(t)}e^{-2t}\,dt \right\}, \quad \dot{x} = u, \quad x(0) = 1, \quad \lim_{t\to+\infty} x(t) \geq 0.$$

(b)

$$\max_{u(t)\in\mathbb{R}} \left\{ \int_0^{+\infty} \log(u(t))e^{-t/5}\,dt \right\}, \quad \dot{x} = x/10 - u, \quad x(0) = 10, \quad \lim_{t\to+\infty} x(t) \geq 0.$$