Foundations of Financial Economics
Two-period DSGE
Arrow-Debreu economies

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Topics

Two period Stochastic General Equilibrium pricing of intertemporal contracts:
to set up a model we need assumptions regarding:

- The economic environment: information tree, real part of the economy
- The market environment: available contracts
- The variables defining the general equilibrium depend on those two categories.

We will study two models:

- Arrow-Debreu economies (the simple case in this lecture, and some extensions later on)
- Finance (or Radner) economies (in next lectures)

Environments and general equilibrium

Common assumptions: regarding the economic environment

1. the time-information structure;
2. the real part of the economy: intertemporal preferences and availability of resources

Different assumptions regarding the market environment

1. simultaneous markets (Arrow-Debreu economy);
2. sequential markets (Finance economy);

Lead to **different definitions of GE** (general equilibrium) (that may be **equivalent or not**)

The time-information tree

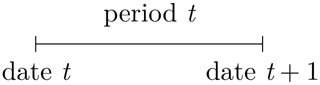
The time-information tree

This refers

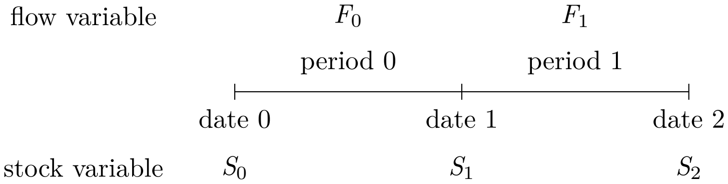
- to the moments in which markets open
- to the timing of the decisions
- the information households have

In discrete time we have to distinguish between

- dates: the timing for **stocks** and prices of stocks
- periods: the timing for **flows** and prices of flows



Two period: The timing for flow and stock variables



Flow and stock variables: refer to prices and/or quantities

For flow variables

We assume:

- $t \in \mathbb{T} = \{0, 1\}$ where $\mathbb{T}$ refer to periods
- information changes over time, from the perspective of period $t = 0$.

Most variables are 2-period random sequences

$$X = \{X_0, X_1\}$$

are determined on the basis of the information known at period $t = 0$:

- at period $t = 0$, they are observed

$$X_0 = x_0$$

- for period $t = 1$, they are contingent on the information available at period $t = 0$

$$X_1(\omega),\ \omega \in (\Omega, \mathcal{F}, \mathbb{P})$$

  X_1 is a random variable

Information for a flow variable

The information at period $t = 0$ is:

- If Ω is discrete and there are N elementary events, the information regarding period $t = 1$ we have

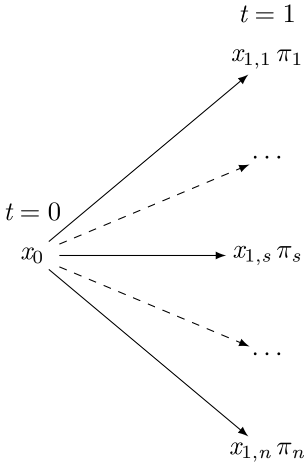
$$X_1 = (x_{1,1}, \dots, x_{1,s}, \dots, x_{1,N})^\top$$
$$P_1 = (\pi_1, \dots, \pi_s, \dots, \pi_N)^\top$$

  where $x_{1,s}$ is the **outcome** if event s realizes and π_s its probability

- and the sequences of possible outcomes and related probabilities are

information	$x_0, 1$	$\begin{pmatrix} x_{1,1} \\ \dots \\ x_{1,s} \\ \dots \\ x_{1,N} \end{pmatrix}, \begin{pmatrix} \pi_1 \\ \dots \\ \pi_s \\ \dots \\ \pi_N \end{pmatrix}$
periods	0	1

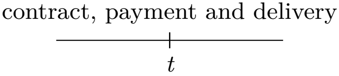
The time-information tree



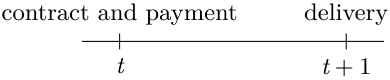
Timing of contracts: for stocks

We distinguish:

- **spot** contracts: contract, delivery and payment done in the same period

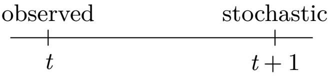


- **intertemporal or forward** contracts: contract and payment in one period, delivery in a future period



They differ along two dimensions:

- the **timing** (which may be relevant if there is , v.g., impatience, depreciation)
- the **information** set associated to the several actions (and prices) involved



Timing of contracts: for flows

- **spot contracts**

  contract, payment and delivery

  t | $t+1$

- **forward contracts**

  contract and payment | delivery

  t | $t+1$

- **information**

  observed | stochastic

  t | $t+1$

The real part of the economy

The real part of the economy

Refers to:

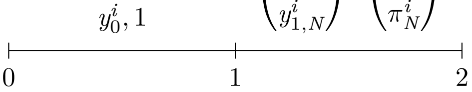
- technology: the type of availability of resources
  - **exchange** economies: the availability of the resources is independent of decisions over time,
  - **production** economies: availability of resources is dependent on decisions in previous periods
- preferences: choice among random sequences of consumption
- distribution of households: they can be **homogenous** or **heterogenous** regarding
  - endowments or technology
  - preferences
  - information

Technology

If we consider a flow of resources for household i:

- The resource for household i is a process $\{Y^i\} = \{y_0^i, Y_1^i\}$ where $y_{t,s}^i$ is the endowment of household i at time t for the state of nature s, with possible realizations and probabilities

$$\begin{pmatrix} y_{1,1}^i \\ \dots \\ y_{1,s}^i \\ \dots \\ y_{1,N}^i \end{pmatrix}, \begin{pmatrix} \pi_1^i \\ \dots \\ \pi_s^i \\ \dots \\ \pi_N^i \end{pmatrix}$$



- in an exchange economy

$$Y_1^i \text{ independent of } y_0^i$$

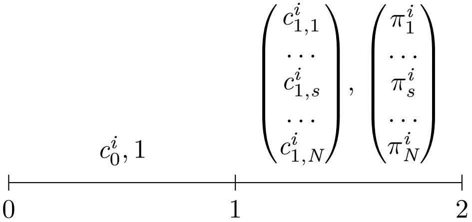
- in a production economy

$$Y_1^i = F_1^i(y_0^i) \text{ dependent on } y_0^i$$

Preferences

household i **chooses** among:

- Sequences of consumption $\{C^i\} = \{c_0^i, C_1^i\}$ is the consumption flow for household i



  where the probabilities can be objective or subjective, exogenous or endogenous, homogeneous or heterogeneous

- Evaluated by an **intertemporal utility functional**

$$U^i(\{C^i\}) = U^i\big(c_0^i, C_1^i\big)$$

Preferences

The two cases have already been considered (see last slide)

- discounted time-additive von-Neumann Morgenstern functional

$$U(\{C\}) = u(c_0) + \beta \mathbb{E}[u(C_1)]$$

- Epstein-Zin utility (see last slide)

$$U(\{C\}) = u^{-1}\left[(1-\beta)\, u(c_0) + \beta\, u\Big(v^{-1}\mathbb{E}[v(C_1)]\Big)\right)$$

Distribution of households

Distribution

- The idiosyncratic components defining a household are:
  - endowments (Y^i)
  - preferences (β^i, u^i) (impatience, risk aversion)
  - information $\mathbb{P}^i$ (only makes sense with subjective probabilities)
- households can be homogeneous or heterogeneous regarding one or all of the previous variables and parameters

  in a homogeneous, or **representative household** economy: endowments, preferences and information are equal

  in a heterogeneous economy: **households differ** in at least one of the three dimensions: endowments ($Y^i \neq Y^j$), preferences ($\beta^i \neq \beta^j$ or $u^i(.) \neq u^j(.)$), or information ($\mathbb{P}^i \neq \mathbb{P}^j$)

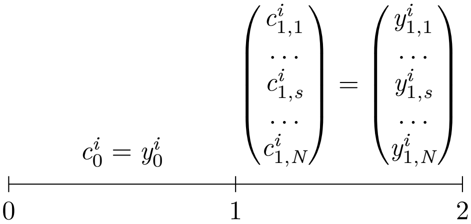
The market structure

Autarky versus trade economies

The economies are distinguished by the exchanges that households can make.

- In autarky all households are hand-in-mouth households

$$c^i_{t,s} = y^i_{t,s},\ t = 0, 1,\ s = 1, \ldots, N$$

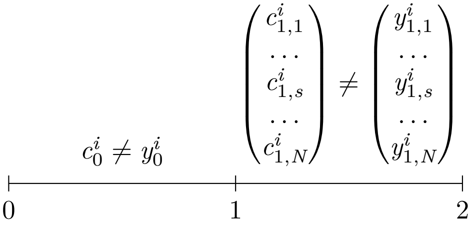


Autarky versus trade economies

- If there are markets for intertemporal transfers of contingent goods, households can trade and be able to make

$$c^i_{t,s} \neq y^i_{t,s},\ t = 0, 1,\ s = 1, \dots, N$$

by shifting resources across **time** (savings) and **states of nature** (self-insurance).



Real versus financial markets

We distinguish further:

- **real markets**:
  market for goods,
  which can be spot or forward
  prices and deliveries are referred to **periods**
- **financial markets**:
  market on financial instruments,
  which are always forward (in an economic sense)
  and prices and deliveries are referred to **dates**

Markets and general equilibrium models

Simultaneous versus sequential market economies

We consider next two economies which are distinguished by the type of intertemporal contracts available:

- **Arrow Debreu economies**:
  there are AD contingent goods traded in spot and forward **real** markets $\Rightarrow$ there is simultaneous market equilibrium
- **finance economies**:
  Radner economies in which **financial** assets are traded $\Rightarrow$ there is sequential market equilibrium

They can be **equivalent under some conditions**, i.e., have the same equilibrium allocations

Two-period DSGE for Arrow-Debreu economies

Summary

Two period Arrow-Debreu exchange economy

1. Contracts and markets
2. The household problem
3. The dynamic stochastic general equilibrium (DSGE) for a general economy
4. The dynamic stochastic general equilibrium (DSGE) for a representative household economy (RAE)
5. Characterizing the DSGE for the RAE

1. Contracts and markets

AD exchange economy: markets

Existing markets:

- 1 spot market operating at period $t = 0$, where the price p_0 is set
- N markets for AD contracts operating at period $t = 0$, where the price vector $\tilde{Q}$ clears the market.

We can **characterize AD markets** by the payoff sequence $\{\tilde{Q}, X_1\}$ where

- prices are

$$\tilde{Q} = (\tilde{q}_1, \ldots, \tilde{q}_s, \ldots, \tilde{q}_N)$$

- and the deliveries are

$$X_1 = (x_{1,s})_{s=1}^{N} = \begin{pmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & 1 \end{pmatrix}$$

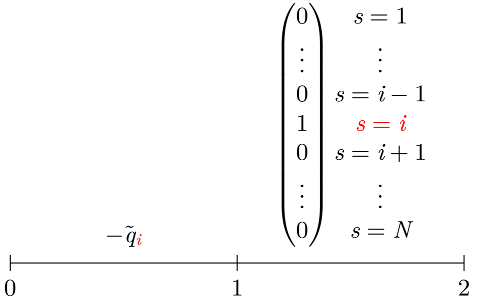
AD exchange economy: Arrow-Debreu contracts

AD contract: is a **real forward contract** such that

- for a price associated to state $s = i$, $\tilde{q}_i$ paid in period $t = 0$
- there is delivery of a contingent good in period $t = 1$ at state $s = i$

$$x_{1,i} = \begin{cases} 1, & \text{if } s = i \\ 0, & \text{if } s \neq i \end{cases}$$

Every contract generates the payoff flow :$\{-\tilde{q}_i, X_{1,i}\}$:



This allows to extend the static GE theory to the present intertemporal and stochastic economy context

AD exchange economy: Arrow-Debreu contracts

Transactions in every AD market:

- The number of contracts is

$$Z_1 = (z_{1,1}, \ldots, z_{1,s}, \ldots, z_{1,N})^\top$$

  where
  - if the household is a buyer of the k-contract, then $z_{1,k} > 0$, and
    - **pays** $\tilde{q}_k z_k$ at $t = 0$
    - **receives** z_k units of the good at $t = 1$ if the state k occurs and 0 otherwise
  - if the household is a seller of the l-contract, then $z_{1,l} < 0$, and
    - **receives** $\tilde{q}_l z_l$ at $t = 0$ and
    - **delivers** z_l units of the good at $t = 1$ if the state l occurs and 0 otherwise
- Then total net expenditure in all AD markets is

$$\tilde{Q}.Z_1 = \sum_{s=1}^{N} \tilde{q}_s z_{1,s} = \underbrace{\sum_{s=1}^{B} \tilde{q}_s z_{1,s}}_{+} + \underbrace{\sum_{s=B+1}^{N} \tilde{q}_s z_{1,s}}_{-}$$

  If it buys the first B contracts and sels $N - B$ contracts

AD exchange economy: transactions

- Transactions in the spot market:

  the net demand: z_0.
  then the total expenditure is $p_0 z_0$
- Transactions in the AD market:

  the net demand: Z_1.
  then the total expenditure is $\tilde{Q}.Z_1$
- The total net expenditure in period $t = 0$ is

$$p_0\, z_0 + \tilde{Q}.Z_1 = 0$$

2. Household's problem

AD exchange economy: consumption financing

- Household i receives a sequence of **endowments**

$$\{Y^i\} = \{y_0^i, Y_1^i\}$$

- Which finance the (random) sequence of consumption, $\{C^i\} = \{c_0^i, C_1^i\}$, out of his endowment, such that
  - in the period $t = 0$

$$c_0^i = z_0^i + y_0^i$$

  - in period $t = 1$, contingent on the information available and contracts done at time $t = 0$

$$C_1^i = Z_1^i + Y_1^i \iff \begin{pmatrix} c_{1,1}^i \\ \dots \\ c_{1,s}^i \\ \dots \\ c_{1,N}^i \end{pmatrix} = \begin{pmatrix} z_1^i \\ \dots \\ z_s^i \\ \dots \\ z_N^i \end{pmatrix} + \begin{pmatrix} y_{1,1}^i \\ \dots \\ y_{1,s}^i \\ \dots \\ y_{1,N}^i \end{pmatrix}$$

AD exchange economy: household's budget constraint

As

$$\begin{cases} c_0^i - y_0^i = z_0^i, & \text{for } t = 0 \\ c_{1,s}^i - y_{1,s}^i = z_{1,s}^i, & \text{for } t = 1, \text{ for every } s = 1, \ldots, N \end{cases}$$

i.e. in every period and for any state of nature **total income is equal to total expenditure**

then the budget constraint at time $t = 0$ (i.e., in the beginning of period 0) is

$$p_0 \left(c_0^i - y_0^i\right) + \tilde{Q}.(C_1^i - Y_1^i) = p_0 \left(c_0^i - y_0^i\right) + \sum_{s=1}^{N} \tilde{q}_s \left(c_{1,s}^i - y_{1,s}^i\right) = 0$$

AD exchange economy: stochastic discount factor

We define:

- the relative price of AD contracts also called the price of the state of nature
$$Q^\top = \Big(q_1, \dots, q_s, \dots, q_N\Big)$$
where
$$q_s \equiv \frac{\tilde{q}_s}{p_0},\ s = 1, \dots, N.$$

- the stochastic discount factor is
$$M^\top = \Big(m_1, \dots, m_s, \dots, m_N\Big)$$
where
$$m_s \equiv \frac{q_s}{\pi_s},\ s = 1, \dots, N.$$

AD exchange economy: household's problem

Choose a **contingent plan** $\{C^i\} = \{c^i_0, C^i_1\}$:

- that maximizes the **intertemporal utility** functional

$$U^i(\{C^i\}) = U^i\Big(c^i_0, C^i_1\Big) = U^i\Big(c^i_0, \big(c^i_{1,1}, \dots, c^i_{1,N}\big)\Big)$$

- subject to the **intertemporal (instantaneous) budget constraint**

$$c^i_0 + \sum_{s=1}^{N} q_s\, c^i_s = y^i_0 + \sum_{s=1}^{N} q_s\, y^i_s$$

- given: the AD prices and endowments $(Q, \{Y^i\})$,

  We define the **wealth of the household** by the value of the endowments at $t = 0$

$$h^i_0 \equiv y^i_0 + \sum_{s=1}^{N} q_s\, y^i_s$$

AD exchange economy: household's problem

- Formally the problem for household i is

$$\max_{c_0^i, C_1^i} U^i\left(c_0^i, C_1^i\right)$$
$$\text{subject to}$$
$$c_0^i + Q \cdot C_1^i = h_0^i$$

- Particular case: If the utility functional is vNM we have

$$\max_{c_0^i, C_1^i} U^i\left(c_0^i, C_1^i\right) = u^i(c_0^i) + \beta\,\mathbb{E}^i[u^i(C_1^i)]$$
$$\text{subject to}$$
$$c_0^i + Q \cdot C_1^i = h_0^i$$

- The index i denotes potential idiosyncratic differences in wealth (h^i), information ($\mathbb{E}^i$), in patience (β^i) and in aversion to risk (u^i)

AD exchange economy: household's problem

Solution for the benchmark case

- Consider the case for any household i (I economize in the notation)
- The Lagrangian

$$\mathcal{L} = u(c_0) + \beta \sum_{s=1}^{N} \pi_s \, u(c_{1,s}) + \lambda \left(h_0 - c_0 - \sum_{s=1}^{N} q_s \, c_{1,s} \right)$$

- The f.o.c are

$$\frac{\partial \mathcal{L}}{\partial c_0} = 0 \iff u'(c_0) = \lambda$$

$$\frac{\partial \mathcal{L}}{\partial c_{1,s}} = 0 \iff \beta \, \pi_s \, u'(c_{1,s}) = \lambda \, q_s \text{ for } s = 1, \dots, N$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \iff h_0 = c_0 + \sum_{s=1}^{N} q_s \, c_{1,s}$$

AD exchange economy: household's problem

Solution for the benchmark case

- At the households's optimum c_0^*, C_1^* we have

$$q_s\, u'(c_0^*) = \beta\, \pi_s\, u'(c_{1,s}^*) \text{ for } s = 1, \ldots, N$$

$$c_0^* + \sum_{s=1}^{N} q_s\, c_{1,s}^* = h_0 = y_0 + \sum_{s=1}^{N} q_s\, y_{1,s}$$

- There are $1 + n$ equations and $1 + n$ variables: although only one state of nature s will be realized, the household has to make sure that it can consume in every possible state of nature (Intuition: our wardrobe - although we have different cloths for different situations we only wear one at a time, and we can only wear any cloth that we bought previously)
- Observe that **the intertemporal marginal rate of substitution, for each state, is equal to the inverse stochastic discount factor**

$$IMRS_{0,1,s} = \frac{u'(c_0^*)}{\beta\, u'(c_{1,s}^*)} = \frac{\pi_s}{q_s} = \frac{1}{m_s} \text{ for } s = 1, \ldots, N$$

3. DSGE: general definition

AD exchange economy: general equilibrium

Definition 1

The DSGE for an endowment AD economy is ***defined*** *by the random sequence of distribution of consumption over time and across households,* $(C^{i,eq})_{i=1}^{I}$*, where* $(C^{i,eq})_{i=1}^{I} = \left(\{c_0^{i,eq}, C_1^{i,eq}\}\right)_{i=1}^{I}$*, and by the AD prices,* Q^{eq}*, given the random sequence of distribution of endowments* $\left(\{y_0^i, Y_1^i\}\right)_{i=1}^{I}$*, such that:*

- *every household* $i \in \mathcal{I}$ *determines the optimal sequence of consumption, taking* Y^i *and* Q *as given, by solving*

$$\{C^{i*}\} = arg\ max\left\{U^i(c_0^i, C_1^i)\ s.t.\ c_0^i + Q \cdot C_1^i \leq h_0^i\right\}$$

- *and markets clear:*

$$\sum_{i=1}^{I} c_0^{i*} = \sum_{i=1}^{I} y_0^i,$$

$$\sum_{i=1}^{I} c_{1,s}^{i*} = \sum_{i=1}^{I} y_{1,s}^i, for\ each\ s = 1, \ldots, N$$

AD general equilibria: intuition

- Allows for the determination:
  - of the Arrow-Debreu price $Q = (q_1, \dots q_N)$: market price for transactions across time and the states of nature
  - or the stochastic discount factor $M = (m_1, \dots m_N)$: defined as $m_s = \dfrac{q_s}{\pi_s}$
- In the types of economy
  - Heterogeneous household economy: dependent upon the preferences, information and the endowments of the economy **and their distribution among households** (i.e, when there are differences in information, attitudes towards risk and wealth)
  - Homogeneous (representative) household economy: dependent upon the preferences, information and the endowments of the economy

4. DSGE: representative household economy

AD exchange and homogeneous economy: general equilibrium

Assume households are homogeneous: same preferences, same information, same endowments

AD exchange and homogeneous economy: general equilibrium

Definition 2

The DSGE for representative household exchange AD economy is ***defined*** *by the sequence of consumption and prices* $(\{c_0^{eq}, C_1^{eq}\}, Q^{eq})$ *such that:*

- *the representative household determines the optimal sequence*

$$C^* = arg\ max\{U(c_0, C_1)\ s.t.\ c_0 + Q \cdot C_1 = h_0\}$$

  given $Y = \{Y_0, Y_1\}$ *and* Q,

- *markets clear*

$$c_0^* = y_0,$$
$$C_1^* = Y_1$$

  or, equivalently

$$c_{t,s}^* = y_{t,s},\ for\ each\ t = 0, 1,\ for\ each\ s = 1, \dots, N$$

AD exchange and homogeneous economy: general equilibrium

Assume:

- households are homogeneous: same preferences, same information, same endowments
- households are characterized by a von-Neumann Morgenstern additive intertemporal utility functional

DSGE RAE with von-Neumann Morgenstern preferences

AD exchange and homogeneous economy: general equilibrium

Definition 3

The DSGE for representative household exchange AD economy is ***defined*** *by the random sequence of consumption and AD-prices* $(\{c_0^{eq}, C_1^{eq}\}, Q^{eq})$ *such that:*

- *the representative household determines the optimal sequence*

$$C^* = arg\ max\{u(c_0) + \beta\, \mathbb{E}_0[u(C_1)]\ s.t.\ c_0 + \mathbb{E}_0\ [MC_1] \leq h_0\}$$

  given $Y = \{Y_0, Y_1\}$ *and* Q*,*

- *markets clear*

$$c_0^* = y_0, \ldots, C_1^* = Y_1$$

  or, equivalently

$$c_{t,s}^* = y_{t,s},\ t = 0, 1,\ s = 1, \ldots, N$$

AD exchange and homogeneous economy: general equilibrium

Observation:

- Defining the stochastic discount factor

$$M = Q \cdot P^{-1} = \left(\frac{q_1}{\pi_1}, \cdots, \frac{q_s}{\pi_s}, \cdots, \frac{q_N}{\pi_N} \right)$$

- we can write the budget constraint

$$c_0 + \sum_{s=1}^{N} q_s \, c_{1,s} = y_0 + \sum_{s=1}^{N} q_s \, y_{1,s}$$

as

$$c_0 + \sum_{s=1}^{N} \pi_s \, m_s \, c_{1,s} = y_0 + \sum_{s=1}^{N} \pi_s \, m_s \, y_{1,s}$$

that is

$$c_0 + \mathbb{E}_0[M \, C_1] = y_0 + \mathbb{E}_0[M \, Y_1]$$

Determination of equilibrium prices

For the benchmark utility functional

The equilibrium is **represented** by the following equations:

1. first, the optimality conditions for the household, assuming there is no satiation $u'(c) > 0$

$$u^{'}(c_0^*)\, q_s = \beta\, \pi_s\, u^{'}(c_{1,s}^*),\ s = 1, \dots, N$$

$$c_0^* + \sum_{s=1}^{N} q_s\, c_{1,s}^* = y_0 + \sum_{s=1}^{N} q_s\, y_{1,s}$$

2. second, the market equilibrium conditions

$$c_0^* = y_0$$

$$c_{1,s}^* = y_{1,s},\ \text{for each } s = 1, \dots, N$$

3. Then, the budget constraint always holds and substituting in the arbitrage conditions the market equilibrium conditions yields,

$$u^{'}(y_0)\, q_s = \beta\, \pi_s\, u^{'}(y_{1,s}),\ s = 1, \dots, N$$

that we can solve for q_s

Equilibrium AD prices and stochastic discount factor

- The **equilibrium AD price** is

$$q_s^{eq} = \beta \pi_s \left(\frac{u^{'}(y_{1,s})}{u^{'}(y_0)} \right), \; s = 1, \ldots, N$$

- or, alternatively, the **equilibrium** stochastic discount factor is

$$m_s^{eq} = \beta \left(\frac{u^{'}(y_{1,s})}{u^{'}(y_0)} \right), \; s = 1, \ldots, N$$

5. DSGE for a representative agent economy (RAE): characterization

AD exchange and homogeneous economy

Proposition 1

Assume an endowment homogenous Arrow-Debreu economy in which the utility functional is a time additive von-Neumann Morgenstern utility functional. Then the DGSE is the sequence of consumption $\{c_0^{eq},\ C_1^{eq}\}$ *and the AD price* Q^{eq} *such that*

$$c_0^{eq} = y_0 \textit{ in period } t = 0$$
$$c_{1,s}^{eq} = y_{1,s} \textit{ in period } t = 1, \textit{ and for each state } s = 1, \ldots, N$$
$$q_s^{eq} = \beta\,\pi_s \left(\frac{u^{'}(y_{1,s})}{u^{'}(y_0)}\right), \textit{ for } s = 1, \ldots, N$$

AD exchange and homogeneous economy

Equilibrium consumption

Then the general equilibrium when households are homogeneous and there is no satiation :

- **no saving and no trade** consumption is equal the endowment (as in the autarkic economy)

$$\{C_t^{eq}\}_{t=0}^{1} = \{Y_t\}_{t=0}^{1}$$

- there is **aggregate uncertainty**: because the endowment Y_1 is stochastic;
- there is **no self insurance**: because, in equilibrium, $C_1^{eq} = Y_1$ consumption is stochastic, (same distribution of consumption and of endowments)

AD exchange and homogeneous economy

Equilibrium AD price

- The equilibrium relative price for AD contracts is also stochastic

$$Q^{eq} = \left(\beta\pi_1 \left(\frac{u^{'}(y_{1,1})}{u^{'}(y_0)} \right), \ldots, \beta\pi_N \left(\frac{u^{'}(y_{1,N})}{u^{'}(y_0)} \right) \right)^{\top}$$

  is a function of the **fundamentals** (resources, preferences and information)

- as $q_s^{eq}(y_0, Y_1)$ if the $u(\cdot)$ is concave

$$\frac{\partial q_s^{eq}}{\partial y_0} > 0, \ \frac{\partial q_s^{eq}}{\partial y_{1,s}} < 0, \ \frac{\partial q_s^{eq}}{\partial y_{1,s'}} = 0$$

  increases with y_0, decreases with $y_{1,s}$ and is neutral for $y_{1,s'}$ (no response to the whole distribution)

- and also

$$\frac{\partial q_s^{eq}}{\partial \beta} > 0, \ \frac{\partial q_s^{eq}}{\partial \pi_s} > 0, \ \frac{\partial q_s^{eq}}{\partial \pi_{s'}} = 0$$

  decreases with patience, increases with the probability of the own state but is neutral to the probabilities of the other states

AD exchange and homogeneous economy

Equilibrium AD price

- The equilibrium stochastic discount factor (SDF)

$$M^{eq} = \left(\beta \left(\frac{u'(y_{1,1})}{u'(y_0)} \right), \ldots, \beta \left(\frac{u'(y_{1,N})}{u'(y_0)} \right) \right)^\top$$

  which is again a function of the **fundamentals** (resources and preferences)

- has the same characterization, but is independent from π_s

$$m_s^{eq} = m_s^{eq}\left(\overset{+}{\beta}, \overset{+}{y_0}, \overset{0}{y_{1,1}}, \ldots, \overset{-}{y_{1,s}}, \ldots, \overset{0}{y_{1,N}} \right)$$

- Interpretation: sign + increases in net demand for future consumption; sign − increase in net future supply; 0 consequence of the independence between states of nature assumption in the vNM utility functional $U(c_0, C_1)$

An example with log utility

SDF for state s

Assuming:

- logarithmic Bernoulli utility function

$$u(c) = \ln(c)$$

- stochastic endowment's growth factor

$$y_{1,s} = (1 + \gamma_s) y_0, \ s = 1, \ldots, N$$

- How does uncertainty affects the stochastic discount factor and the utility of the household ?

An example with log utility

Distribution of the SDF

- the stochastic discount factor is $m_s^* = \frac{\beta}{1+\gamma_s}$

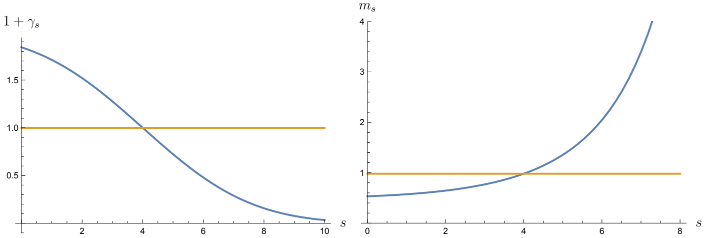


Figure: Growth factor $(1+\Gamma)$ and stochastic the associated discount factor M

- Conclusions:
  1. there is **aggregate uncertainty**
  2. stochastic discount factor is **negatively correlated** with the anticipated rate of growth

An example with log utility

Sampling the SDF

- the stochastic discount factor is

$$m_s^{eq} = \frac{\beta}{1+\gamma_s}$$

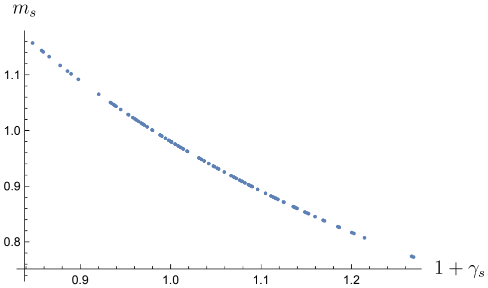


Figure: Sampling from $\gamma \sim N(0.05, 0.1)$ and the stochastic discount factor

An example with log utility

Aggregate uncertainty and lack of insurance

- The utility for the household is (prove it)

$$
\begin{aligned}
U(C^*) &= \ln(c_0^*) + \beta \mathbb{E}_0[\ln(C_1^*)] = \\
&= \ln(y_0) + \beta \mathbb{E}_0[\ln(Y_1)] = \\
&= \ln\left(y_0^{1+\beta} \left(G \mathbb{E}_0[1+\Gamma] \right)^{\beta} \right)
\end{aligned}
$$

  increases with y_0 and with the geometric mean of the growth rate.
- Question: why this looks like the utility in a Robinson-Crusoe economy ?
- Question: what are the consequences of more volatility, to the stochastic discount factor and to household's utility ?

DSGE RAE with EZ preferences

AD exchange and homogeneous economy

Epstein-Zin preferences

- DSGE representation

$$
\begin{aligned}
q_s\, U_0(c_0, C_1) &= U_{1s}(c_0, C_1), \text{ for } s = 1, \dots, N \\
c_0 &= y_0 \\
c_{1,s} &= y_{1,s}, \text{ for } s = 1, \dots, N
\end{aligned}
$$

- the equilibrium price is

$$q_s^{eq} = \frac{U_{1s}(y_0, Y_1)}{U_0(y_0, Y_1)}, \text{ for } s = 1, \dots, N$$

- using our previous slide we have

$$m_s^{eq} = \beta\, \mathbb{E}[\, Y_1^{1-\varrho}]^{\frac{\varrho-\zeta}{1-\varrho}}\, y_{1,s}^{-\varrho}\, y_0^{\zeta}$$

- Setting $y_{1,s} = (1+\gamma_s)\, y_0$

$$m_s^{eq} = \beta\, \mathbb{E}[(1+\Gamma)^{1-\varrho}]^{\frac{\varrho-\zeta}{1-\varrho}}\, (1+\gamma_s)^{-\varrho}$$

AD exchange and homogeneous economy

Epstein-Zin preferences

- differently from the vNM case, the stochastic discount factor

$$m_s^{eq} = m_s^{eq}\big(\overset{+}{\beta}, \overset{+}{y_0}, \overset{?}{y_{1,1}}, \dots, \overset{?}{y_{1,s}}, \dots, \overset{?}{y_{1,N}}\big), \text{ for } s = 1, \dots, N$$

  - depends on the **information referring to all the states of nature** from Y_1, but we expect that $\dfrac{\partial m_s}{\partial y_{1,s'}} > 0$ (if $\varrho > \zeta$)
  - depends on the **information referring to all the states of nature** from $\mathbb{P}$, but we expect that $\dfrac{\partial m_s}{\partial \pi_s} > 0$ (if $\varrho > \zeta$) for any s
  - the dependence on the own state is a function of ϱ and not ζ (which determines the level in a negative way)
  - the parameter ζ affects the level of the whole distribution of M

Benchmark and EZ preferences

Sampling the SDF

- the stochastic discount factors for the benchmark and the EZ preferences

$$m_s = \beta\,(1+\gamma_s)^{-\zeta}, \text{ and } m_s = \beta\,\mathbb{E}[(1+\Gamma)^{1-\varrho}]^{\frac{\varrho-\zeta}{1-\varrho}}\,(1+\gamma_s)^{-\varrho}$$

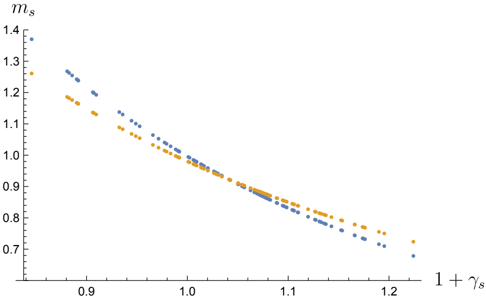


Figure: Sampling from $\gamma \sim N(0.05, 0.1)$ and for $\zeta = 1.5$ and $\varrho = 1.9$ and the stochastic discount factor (blue: EZ, brown: benchmark)

References

- (LeRoy and Werner, 2014, Part III), (Lengwiler, 2004, ch. 2), (Altug and Labadie, 2008, ch. 3)

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