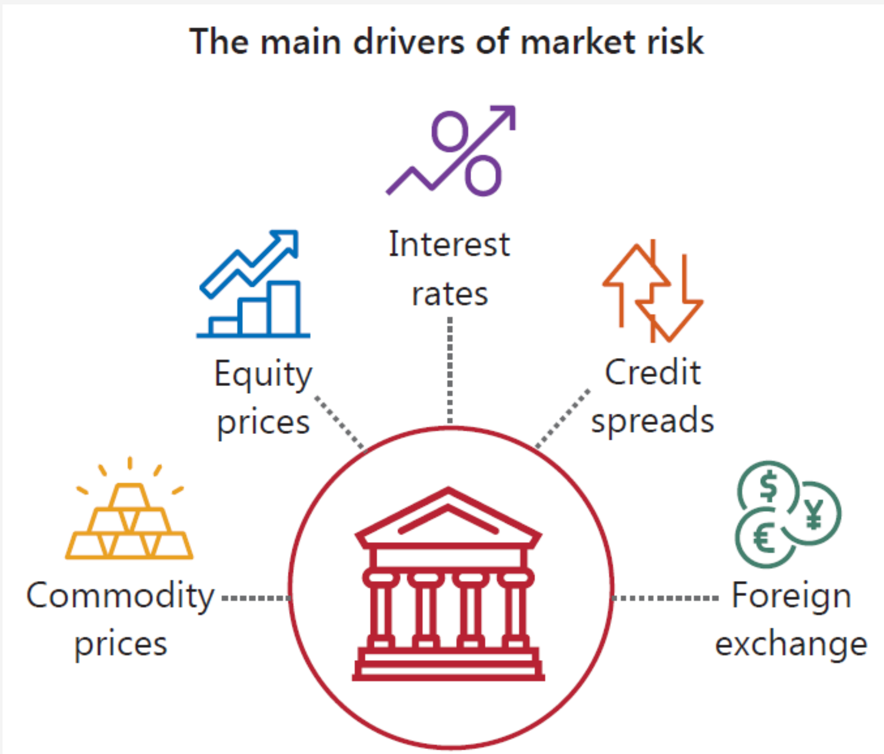
4. Market Risk

Market Risk

- Definition: risk of losses due to the impact of interest rate, exchange rates, stock and bond prices or other financial asset price moves on the value of actively traded portfolios.



Source: BIS (2019), “The Market Risk Framework – in brief”.

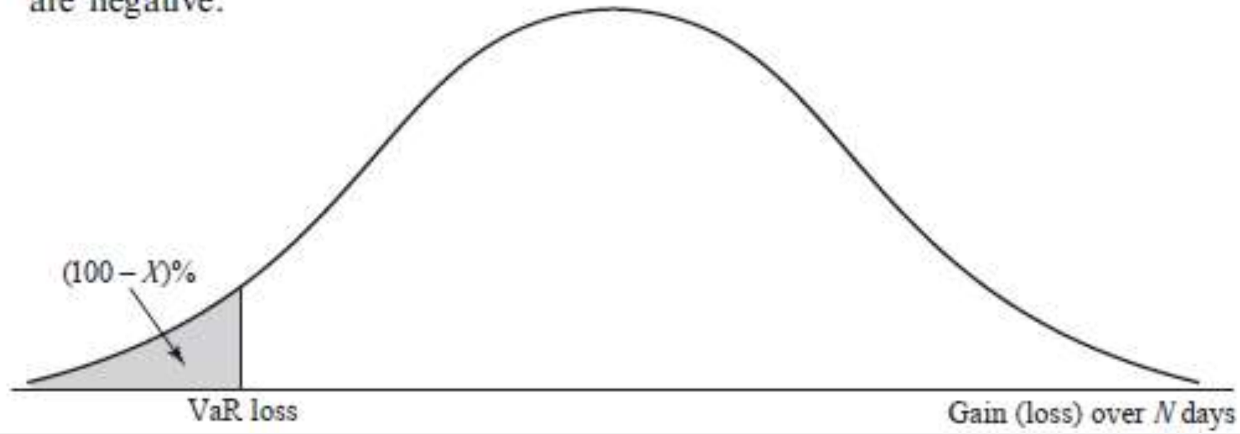
4.1. Value-at-Risk (VaR)

VaR

Please see Jorion (2007) – Chapters 1 and 2

- Usual measure of the market risk of a portfolio: Value-at-Risk (VaR).
- VaR: maximum loss that can occur with *X*% confidence over a holding period of *t* days, being *X* the confidence level (usually high) and *t* a short number of days, providing a conservative loss measure, corresponding to a highly unlikely but severe scenario.

Figure 22.1 Calculation of VaR from the probability distribution of the change in the portfolio value; confidence level is *X*%. Gains in portfolio value are positive; losses are negative.



Source: Hull, John (2018)

VaR

- VaR is a measure of unexpected loss that answers to the following question:

how bad things can get for a financial portfolio comprising different types of financial assets under a set of assumptions?

VAR summarizes the worst loss over a target horizon that will not be exceeded with a given level of confidence.

Source: Jorion (2007)

- While valuation models focus on the mean of the distribution, VaR gives us the potential variation in prices or returns under very unlikely scenarios, being a summary statistic of the probability density function.

VaR

- For a 5% significance level and a daily horizon, 95 out of 100 days the portfolio won't loose more than the VaR.

↓

- With a confidence level of 1 – *X* (i.e. 1-0,05=95%), the loss in a given time horizon is not expected to exceed the VaR.

↓

- To measure VaR we need to define:
  - Confidence level
  - Time horizon
  - Distribution function

VaR

- **Confidence level**
  - Subjective decision
  - Basel Committee uses a 99% level to compute capital requirements for market risk of trading (marked-to-market) portfolios.
- **Time Horizon**
  - Depends on the portfolio strategy and liquidity
  - Usually daily, weekly or monthly
  - Basel Committee uses a 10-day horizon
  - Rule of thumb: portfolios with higher turnover must use shorter time horizons.
- **Distribution**
  - Can be based on an empirical or a parametric approach.

VaR methods

Please see Jorion (2007), Chapters 4 and 5

(i) Empirical or historical approach

(ii) Parametric approach

- The empirical approach does not assume a theoretical distribution a priori for returns, contrary to the parametric approach, as it uses the past data in a very straightforward way, assuming that past variations replicate in the future.
- It corresponds to generate a number of scenarios for all asset prices included in the portfolio, assuming that each scenario is characterized by the variations in each sample day (being v_i the financial asset value in day i, used to estimate the value in day $n+1$ under the ith scenario):

$$\text{Value under } i\text{th scenario} = v_n \frac{v_i}{v_{i-1}}$$

VaR

- Even though VaR provides a conservative loss measure, it doesn't give us the worst potential loss =>

(i) VaR must be complemented other risk measurement tools, e.g. stress tests or Expected Shortfall (ES);

(ii) Exposure or risk limits based on VaR are not enough and must be added by quantitative limits, according to the risk appetite of the portfolio owner, related to the capital he is willing or able to lose.

- Nonetheless, Basel Committee set capital requirements for market risk based on VaR since 1996, as a conservative multiple (k) of the 10-day 99% VaR, being k chosen on a bank-by-bank basis by regulators, with a minimum of 3.

VaR

- **VaR Advantages**
  - It’s a single number, easily accessible and understood
  - Allows for comparison between different products and strategies
  - Enhances performance evaluation and the settlement of risk limits
- **VaR Disadvantages**
  - NOT a loss forecast
  - NOT the worst case scenario
  - NOT fully objective (depends on time horizon and a)
  - NOT the ultimate truth (one may be using the wrong distribution or the wrong period to estimate the parameters)
  - Only works for liquid securities and continuous payoffs
  - Ignores Black Swans

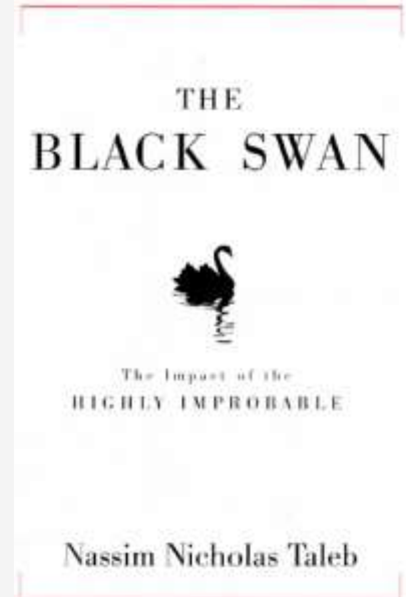
Black Swans

- **Definition**:

(i) An outlier, something completely unexpected according to the past;

(ii) Has an extreme impact (the "Extremistan");

(iii) Even though it is an outlier, economic agents try to find rational explanations for it afterwards, in order to make it predictable in the future.



- **Consequences**:

(i) Being unpredictable, we need to adjust to their existence, instead of trying to predict them;

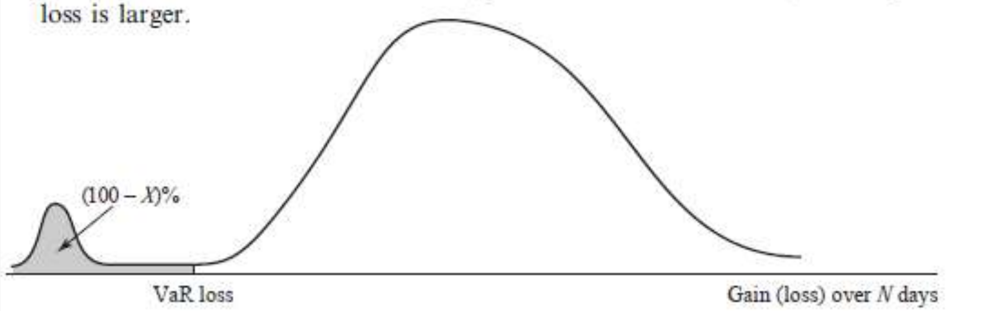
(ii) Therefore, VaR is not a crystal ball, but just a quantitative tool.

4.2. Expected Shortfall

ES

- Identical VaRs can mean different levels of risk, namely if we have multimodal distributions.

Figure 22.2 Alternative situation to Figure 22.1. VaR is the same, but the potential loss is larger.



Source: Hull, John (2018)

- A measure that deals with this problem is ES.
- While VaR asks the question ''How bad can things get?'', ES asks:

''If things do get bad, how much can the company expect to lose?''

- Definition: expected loss during an N-day period conditional on the loss being worse than the VaR.
- Example: with a 99% 10d VaR, ES = average loss over a 10d period when the loss is worse than the 10-day 99% VaR.

4.3. Capital Requirements

Capital Requirements for Market Risk

- **Quantitative Requirements to use VaR:**

(i) Daily calculation

(ii) 99%, 10-day period VaR

(iii) Minimum sample of 1 year, except when higher price volatility justifies a shorter period

(iv) Minimum monthly data update

(v) Minimum weekly frequency for stressed VaR

(vi) VaR is scaled up by a multiplication factor = 3 + additional factor (addend) between 0 and 1, depending on the number of loss excesses observed in the previous 250 business days.

Number of overshootings	addend
Fewer than 5	0,00
5	0,40
6	0,50
7	0,65
8	0,75
9	0,85
10 or more	1,00

Source: European Parliament (2013), CRR.

Capital Requirements for Market Risk

- **Qualitative requirements:**

- Models integrated in bank's daily risk management and internal reports to top management;
- Risk control unit independent from trading and reporting directly to top management, liable for the development, implementation and validation of internal models, producing and analyzing daily reports on model results and presenting proposals on trading limits;
- Board and top management actively involved in risk control processes and daily reports;
- Adequate human resources in trading, risk control, auditing and back-office;
- Internal models with good track record;
- Stress tests - Rigorous and frequent program, with reverse stress tests;
- Internal independent auditing process;
- Minimum yearly internal assessment of the global risk management system.

4.4. Parametric and non-parametric methods of VaR estimation

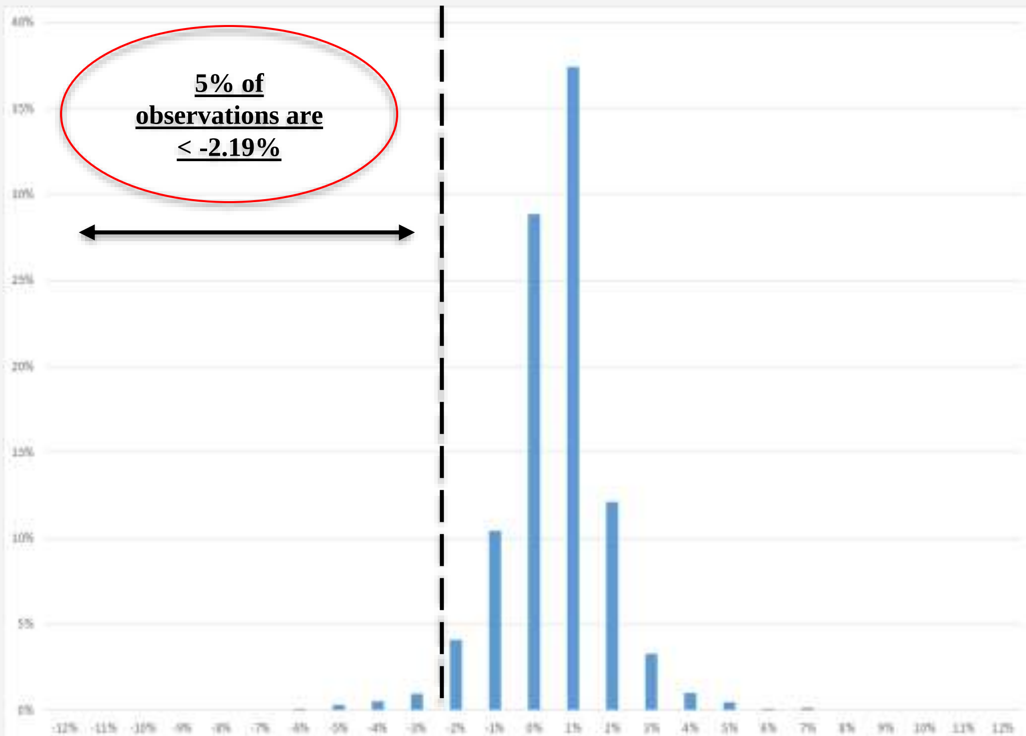
VaR methods

- Generally, the calculation of VaR uses a histogram of the changes in the value of the portfolio (i.e. empirical distribution) for a given pre-defined time horizon and a given α%.
- The higher the volatility, degree of confidence and maturity, the higher will be the VaR.
- The usual assumption is: $N\text{-day VaR} = 1\text{-day VaR} \times \sqrt{N}$
- This assumption is based on the returns being normally distributed and independent => variances are additive over time => volatility grows with the square root of time.
- As the volatility fluctuates along time, the VaR will also change, even when calculated under the same assumptions.
- Volatility also assumes different magnitudes for different classes of financial assets.
- In a portfolio, negative correlations may contribute to mitigate the aggregate volatility.
- VaR can be computed by non-parametric (empirical/historical) or parametric approaches.

VaR methods

(i) Non-parametric or Empirical/Historical approach:

- Empirical distribution of the daily variations of the Nasdaq Index (2705 observations):



- 5% critical level = 135th smallest observation

VaR methods

- 5% critical level = 135th (5% x 2705) smallest observation

- $R^*_{95\%}$ = -2,19% is the daily return for this observation
- μ (average return or average daily growth rate of the index in the full sample) = 0,038%.
- V = 1M€ => daily VaR @ 95% confidence level is:

$$VaR_{95\%}(mean) = -V[\exp(R^*_{95\%}) - \exp(\mu)] = 22.050{,}45€$$

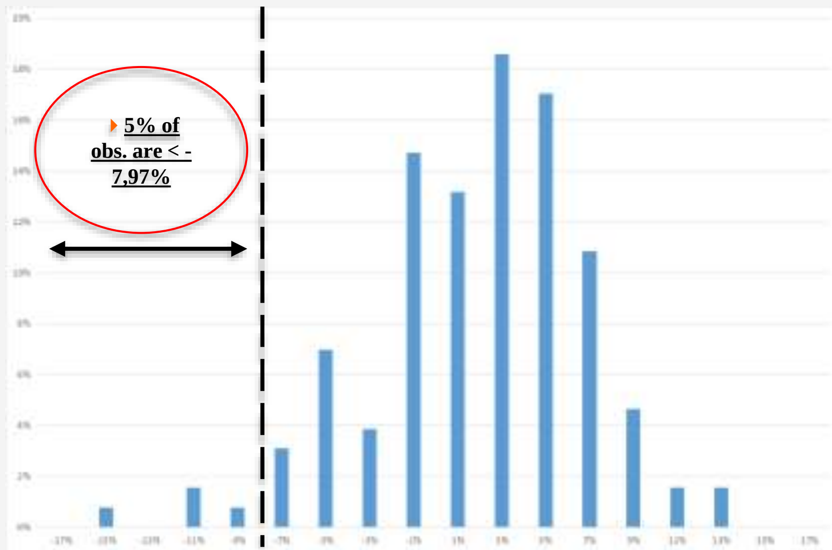
$$VaR\ (current)@\ 95\% = -V[\exp(R^*_{95})] = 21.675{,}25€$$

- **Conclusions:**
  (i) 95 in 100 days the portfolio won't lose more than $22.050,45 during a 1-day period comparing to the expected return, according to the empirical distribution;
  (ii) Comparing to the current value of the portfolio, the estimated loss is $21.675,25 (very similar given that the estimated loss is small, as the period considered is also very short).

VaR methods

- Instead of assuming the usual assumption for the calculation of the VaR for a larger period (by multiplying the shorter-period VaR by the square root of time), one may use the same data to get the empirical distribution for larger horizons.

- For monthly variations of the Nasdaq Index, the empirical distribution (129 observations, assuming non-overlapping periods with 21 days per month) is as follows:



VaR methods

- 5% critical level = 6th smallest observation (5% x 129)

  R* = -7,97%

- Being μ = 0,79% and V = 1M€ => monthly VaR @ 95% confidence level:
  $VaR\ (mean)\ @\ 95\% = -V[\exp(R^*) - \exp(\mu)] = 84.546{,}87€$

- In 95 out of 100 days, the portfolio won't lose more than $84.546,87, according to our empirical distribution.

- Following the usual assumption $N\text{-day VaR} = 1\text{-day VaR} \times \sqrt{N}$, by assuming the normality of returns, using the daily VaR (mean) previously computed (22.050,45€), one would have obtained 22.050,45 x sqrt (21) = $101.047,874.

- This difference to the monthly VaR suggests that returns are not normally distributed.

VaR methods

(ii) Parametric approach

- The main alternative to historical simulation is the model building or parametric approach.
- Most common implementation of VaR assumes that returns follow a normal distribution and are i.i.d.

- Returns are normally distributed ⇔ prices follow a log-normal distribution.

- This method also assumes that:

(i) the correlations between risk factors are constant and the delta (or price sensitivity to changes in a risk factor) of each portfolio asset is constant;

(ii) the volatility of each risk factor is extracted from the historical observation period.

VaR methods

- The simplest parametric method to calculate VaR is based on the assumption of normally distributed daily returns - delta-normal or variance-covariance method:

  $$VaR = \omega' \Sigma \omega \times N^{-1}(X) \times \sqrt{T}$$

  where Σ is the variance-covariance matrix of the portfolio's assets and ω corresponds to the weights of each asset in the portfolio.

- If the portfolio has only 1 asset, VaR results only from that asset volatility:

  $$VaR = \sigma \times N^{-1}(X) \times \sqrt{T}$$

NASDAQ

	Daily	Monthly
σ	1,37%	5,19%
z_{α}	-1,645	-1,645
VaR (current) @ 95%	-2,26%	-8,53%
VaR (current) @ 95%	$22 578	$85 299

Volatilities and Correlations

- **Different methods to calculate relevant risk factor volatilities and correlations:**

(i) Simple historic volatility and correlation - the most straightforward method but the effects of a large single market move can significantly change volatilities and correlations over the required forecasting period, as all observations are equally weighted.

(ii) Weighted historical volatility or correlation - this is done to give more weight to recent observations so that large jumps in volatility are not caused by events that occurred some time ago, using 2 main methods.

(1) Exponentially weighted moving averages - the weights are attached according to an exponential function.

(2) Generalized Autoregressive Conditional Heteroscedasticity (*GARCH*) models.

4.5. VaR for stock portfolios

VaR in a Portfolio of Stocks

- Portfolio algebra:
  - Single-asset and 2-asset cases (Hull (2018), chapter 22)
  - Multi-assets (Jorion (2007), chapter 7)
- VaR in a portfolio of stocks (Jorion (2007), chapter 8):
  - Market model
  - Beta model
  - Factor model

VaR in a Portfolio of Stocks

- Single-asset case
  - assumptions:
    - Exposure to Microsoft shares (*V*): $10M
    - Time horizon (*n*) =10
    - Degree of confidence = 99% => N^{-1} (99%) = 2,326.
    - Daily volatility of returns (σ) = 2% (2%*sqrt(252) = 32%/year) => for this one-asset portfolio 2% x V = $ 200.000
    - Expected change of returns = 0% (reasonable assumption, as the time period is very short and the expected change is much smaller than volatility, e.g. if the annual expected return =20%, the 1-day expected return is 0,2/252 = 0,08%, vs daily volatility = 2%)
- VaR:
    - 1d: N^{-1}(99%) x σ x *V* = 2,326 x $200.000 = $465.300
    - 10d: 1d VaR x sqrt(10) = $1.471.300

VaR in a Portfolio of Stocks

- Single-asset case

  - assumptions:
    - Exposure to AT&T shares (V): $5M
    - Time horizon (n) =10
    - Degree of confidence = 99% => N^{-1} (99%) = 2,326.
    - Daily volatility of returns (σ) = 1% (1%*sqrt(252) = 16%/year) => for this one-asset portfolio 1% x V = $ 50.000
    - Expected change of returns = 0%

  - VaR:
    - 1-day: N^{-1}(99%) x σ x V = 2,326 x $50.000 = $116.300
    - 10-day: 1-day VaR x sqrt(10) = $367.800

VaR in a Portfolio of Stocks

- Two-assets case:

  - Standard-deviation of the portfolio (X and Y correspond to Microsoft and AT&T, respectively):

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

  - Assumption: Correlation between asset returns (ρ) = 30%

  - 1- day standard-deviation of the portfolio:

$$\sqrt{200{,}000^2 + 50{,}000^2 + 2 \times 0.3 \times 200{,}000 \times 50{,}000} = 220{,}200$$

  - 1d VaR: $220.200 x 2,326 = $512.300
  - 10d VaR: $512.300 x sqrt (10) = $ 1.620.100

(< sum of single-asset 10-day VaRs = $1.471.300 + $367.800 = $1.839.100)

VaR in a Portfolio of Stocks

- **Two-assets case with perfect correlation:**

  - Diversification benefits: \$1.839.100 - \$1.620.100 = \$219.000

    ρ = 1 => 1-day standard-deviation =
    $\sqrt{200000^2 + 50000^2 + 2 \cdot 1 \cdot 200000 \cdot 50000} = \$250000$

  - 1-day VaR: $\$250000 \cdot 2{,}326 = 581587$
  - 10-day VaR: $\$250000 \cdot 2{,}326 \cdot \sqrt{10} = 1839139$

  - Sum of single-asset 10-day VaRs = \$1.471.300 + \$367.800 = \$1.839.100

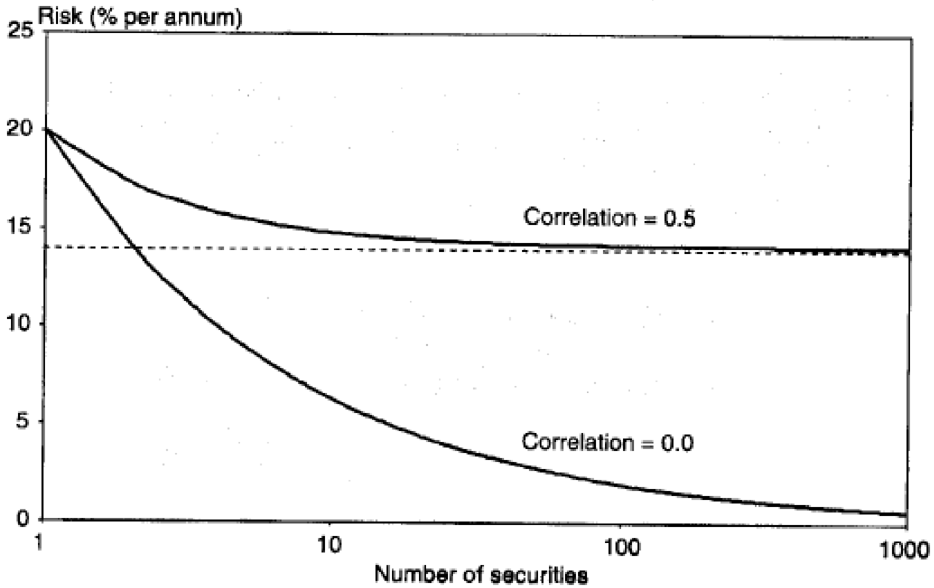
  - Diversification benefits: \$1.839.100 - \$1.839.100 = \$0

VaR in a Portfolio of Stocks

- Therefore, diversification benefits get larger when:
  - the number of securities increase; and
  - the correlation between these returns decreases.

FIGURE 7-1

Risk and number of securities.



$$\sigma_p = \sigma\sqrt{\frac{1}{N} + \left(1 - \frac{1}{N}\right)\rho}$$

Source: Jorion (2007), Chapter 7

VaR in a Portfolio of Stocks

- **Portfolio return – weighted average of returns:**

$$R_{p,t+1} = \sum_{i=1}^{N} w_i R_{i,t+1}$$

where N is the number of assets, $R_{i,t+1}$ is the rate of return on asset i, and w_i is the weight. The *rate of return* is defined as the change in the dollar value, or dollar return, scaled by the initial investment. This is a unitless measure.

- **Matrix notation:**

$$R_p = w_1 R_1 + w_2 R_2 + \cdots + w_N R_N = [w_1 w_2 \cdots w_N] \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix} = w' R$$

VaR in a Portfolio of Stocks

- Portfolio expected return - weighted average of returns:

$$E(R_p)=\mu_p = \sum_{i=1}^{N} w_i \mu_i$$

- Variance of portfolio returns - includes not only the risk of individual assets, but also their covariances:

$$V(R_p)=\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} w_i w_j \sigma_{ij} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \sigma_{ij}$$

Covariance term

This sum accounts not only for the risk of the individual securities σ_i^2 but also for all covariances, which add up to a total of $N(N-1)/2$ different terms.

VaR in a Portfolio of Stocks

- With the total number of assets increasing, one needs to rely on matrix notation:

$$\sigma_p^2 = [w_1 \cdots w_N] \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1N} \\ \vdots & & & & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \cdots & \sigma_N^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \leftrightarrow \quad \sigma_p^2 = w' \Sigma w$$

being Σ the variance-covariance matrix

$$\begin{bmatrix} \text{var}_1 & \text{cov}_{12} & \text{cov}_{13} & \cdots & \text{cov}_{1n} \\ \text{cov}_{21} & \text{var}_2 & \text{cov}_{23} & \cdots & \text{cov}_{2n} \\ \text{cov}_{31} & \text{cov}_{32} & \text{var}_3 & \cdots & \text{cov}_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{cov}_{n1} & \text{cov}_{n2} & \text{cov}_{n3} & \cdots & \text{var}_n \end{bmatrix}$$

being $\quad \text{cov}_{ij} = \sigma_i \sigma_j \rho_{ij}$

VaR in a Portfolio of Stocks

- Matrix notation in monetary units:

$$\sigma_p^2 W^2 = x' \Sigma x$$

being W the portfolio total market value

- Portfolio VaR (being α the N^{-1} for the confidence level):

$$\text{Portfolio VAR} = \text{VAR}_p = \alpha \sigma_p W = \alpha \sqrt{x' \Sigma x}$$

- Obviously, the portfolio VaR can also be calculated straight from the volatility of the aggregate returns.

$$\Downarrow$$

- VaR can be obtained just by computing the standard-deviation of portfolio returns.
- If all asset returns are independent, portfolio VaR is just the sum of all individual VaRs.
- Otherwise, the portfolio VaR must be lower than the sum of all individual VaRs.

VaR in a Portfolio of Stocks

- Returning to the 2-assets example, the portfolio variance is:

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2$$

- Portfolio VaR: $$\mathrm{VAR}_p = \alpha\sigma_p W = \alpha\sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2}\,W$$

- ρ = 0 => $$\mathrm{VAR}_p = \sqrt{\alpha^2 w_1^2 W^2\sigma_1^2 + \alpha^2 w_2^2 W^2\sigma_2^2} = \sqrt{\mathrm{VAR}_1^2 + \mathrm{VAR}_2^2}$$

- ρ = 1 => $$\mathrm{VAR}_p = \sqrt{\mathrm{VAR}_1^2 + \mathrm{VAR}_2^2 + 2\mathrm{VAR}_1 \times \mathrm{VAR}_2} = \mathrm{VAR}_1 + \mathrm{VAR}_2$$

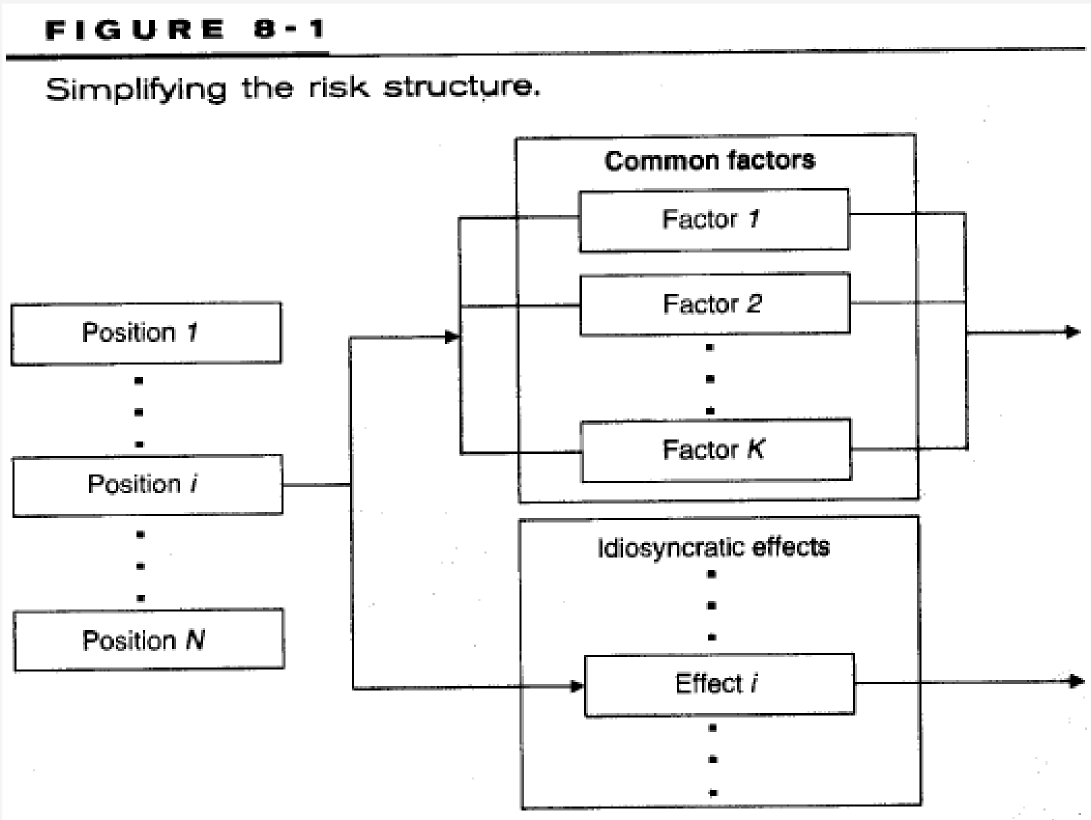
- When the correlation between assets is perfect, the portfolio VaR is the sum of the individual asset VaRs.

VaR in a Portfolio of Stocks

- VaR is a large-scale risk measure, able to aggregate high volumes of data.
- However, when portfolios include a very large numbers of assets, it becomes difficult or even unnecessary to model all exposures individually as risk factors.
- When the number of assets (n) is not too large, the variance-covariance measure demands the estimated of a low number of parameters (n*(n+1)/2).
- However, the number of parameters to be estimated increases with the sum of the number of added assets:
  - 10 assets => 55 parameters to be estimated.
  - 20 assets => 210 parameters (55+11+12+13+…+20).

VaR in a Portfolio of Stocks

- The risk structure of a portfolio can be summarized by a set of common and idiosyncratic factors:



Source: Jorion (2007)

VaR in a Portfolio of Stocks

- The problem of having a too large number of assets in a portfolio may be simplified by using simpler structures for the covariance matrix, e.g. assuming all pairs of assets have the same correlation coefficient (homogeneous correlations).

- Another simple model is the diagonal model, proposed by Sharpe - considers that the returns of stocks are determined by a common factor: the market return.

- The stock return is thus determined by a market return (R_m) and specific term ε_i not correlated with the market and other stocks.

$$R_i = \alpha_i + \beta_i R_m + \epsilon_i$$

General market risk

Specific risk

- Assumptions: $E(\epsilon_i) = 0 \qquad E(\epsilon_i R_m) = 0 \qquad E(\epsilon_i \epsilon_j) = 0$
  - The errors are uncorrelated with the common factor and across each other.
  - β_i = exposure to market or factor loading, being the systematic risk when the market return is represented by the stock market index.

VaR – Diagonal model

- Variances: $\sigma_i^2 = V(R_i^2)$, $\sigma_m^2 = V(R_m^2)$, and $V(\epsilon_i^2) = \sigma_{\epsilon,i}^2$
- **The variance of stock i can be decomposed into systematic and specific risk:**

$$\sigma_i^2 = V(\beta_i R_m + \epsilon_i) = \beta_i^2\sigma_m^2 + 2\,\mathrm{cov}(\beta_i R_m, \epsilon_i) + V(\epsilon_i)$$
$$= \beta_i^2\sigma_m^2 + \sigma^2_{\epsilon,i}$$

- Covariance between 2 assets i and j (as the asset returns are only correlated to the market): $\sigma_{i,j} = \mathrm{cov}(\beta_i R_m + \epsilon_i,\ \beta_j R_m + \epsilon_j) = \beta_i\beta_j\sigma_m^2$
- **Variance-Covariance matrix:**

$$\Sigma = \begin{bmatrix} \beta_1\beta_1\sigma_m^2 + \sigma_{\varepsilon,1}^2 & \cdots & \beta_1\beta_N\sigma_m^2 \\ \vdots & & \vdots \\ \beta_N\beta_1\sigma_m^2 & \cdots & \beta_N\beta_N\sigma_m^2 + \sigma_{\varepsilon,N}^2 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} [\beta_1 \cdots \beta_N]\sigma_m^2 + \begin{bmatrix} \sigma_{\varepsilon,1}^2 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & \sigma_{\varepsilon,N}^2 \end{bmatrix}$$

- Matrix notation: $\Sigma = \beta\beta'\sigma_m^2 + D_\epsilon$
- As matrix D is diagonal, the number of parameters is reduced from n x $(n+1)/2$ to $2n+1$ (N for the betas, N for matrix D and 1 for σ^2).

VaR – Diagonal model

Example:

- $ 1M Equal weight portfolio of the 10 largest caps of Nasdaq Index
- Inputs for VaR modelling – individual stock returns and VCV matrix
- Parameters for VaR modelling - Confidence level (95%) and Time horizon (1 day)

Notional (in Usd)	1 000 000
Confidence Level	95%
Horizon (in days)	1

$$\sigma_i^2 = V(\beta_i R_m + \epsilon_i) = \beta_i^2 \sigma_m^2 + 2\,\mathrm{cov}(\beta_i R_m, \epsilon_i) + V(\epsilon_i)$$
$$= \beta_i^2 \sigma_m^2 + \sigma^2_{\epsilon,i}$$

	Beta	σ^2_i	$\beta^2 . \sigma^2_{mkt}$	σ^2_ε	Daily σ	Daily specific risk	w
MSFT US Equity	0,87	0,0003	0,0001	0,0001	1,66%	1,16%	0,1
AAPL US Equity	1,12	0,0005	0,0002	0,0003	2,28%	1,68%	0,1
YHOO US Equity	0,98	0,0006	0,0002	0,0005	2,53%	2,14%	0,1
ORCL US Equity	0,98	0,0003	0,0002	0,0002	1,85%	1,26%	0,1
CSCO US Equity	1,03	0,0004	0,0002	0,0002	1,95%	1,35%	0,1
NFLX US Equity	0,94	0,0014	0,0002	0,0013	3,79%	3,56%	0,1
AMZN US Equity	1,22	0,0007	0,0003	0,0004	2,67%	2,08%	0,1
QCOM US Equity	1,04	0,0004	0,0002	0,0002	1,96%	1,35%	0,1
AMD US Equity	1,41	0,0012	0,0004	0,0009	3,52%	2,93%	0,1
INTC US Equity	1,02	0,0004	0,0002	0,0002	1,87%	1,24%	0,1
Nasdaq	1,00	0,0002					
Portfolio	**1,06**	**0,0003**	**0,0002**	**0,00004**	**1,59%**	**0,64%**	1

Daily portfolio σ	1,59%	FALSE
1d VaR @ 95%	**26 207**	

VaR – Beta model

- We can use this simplification to compute the risk of a portfolio:

$$R_p = \sum_{i=1}^{N} w_i R_i = w' R$$

- From $\Sigma = \beta\beta'\sigma_m^2 + D_\epsilon$ we get:

$$V(R_p) = V(w'R) = w'\Sigma w = w'(\beta\beta'\sigma_m^2 + D_\epsilon)w = (w'\beta)(\beta' w)\sigma_m^2 + w' D_\epsilon w$$

- When N increases and the portfolio is adequately diversified, the last term of the previous equation tends to zero, as specific risk may be assumed as zero and the risk of the portfolio becomes dominated by the common factor.

$$\Downarrow$$

$$V(R_p) \rightarrow (w'\beta\beta' w)\,\sigma_m^2 = (\beta_p \sigma_m)^2$$

$$\Downarrow$$

- The portfolio risk will be proportional to the market index risk (beta mapping) => the beta model is just a particular case, a restriction, of the diagonal model.

VaR – Beta model

- The VaR is lower than in the diagonal model, as the specific risk is assumed to have been eliminated:

Notional (in Usd)	1 000 000
Confidence Level	95%
Horizon (in days)	1

$$\sigma_i^2 = V(\beta_i R_m + \epsilon_i) = \beta_i^2 \sigma_m^2 + 2\,\mathrm{cov}(\beta_i R_m, \epsilon_i) + V(\epsilon_i)$$
$$= \beta_i^2 \sigma_m^2 + \sigma^2_{\epsilon,i}$$

	Beta	σ^2_i	$\beta^2 . \sigma^2_{mkt}$	σ^2_ε	Daily σ	Daily specific risk	w
MSFT US Equity	0,87	0,0003	0,0001	0,0001	1,66%	1,16%	0,1
AAPL US Equity	1,12	0,0005	0,0002	0,0003	2,28%	1,68%	0,1
YHOO US Equity	0,98	0,0006	0,0002	0,0005	2,53%	2,14%	0,1
ORCL US Equity	0,98	0,0003	0,0002	0,0002	1,85%	1,26%	0,1
CSCO US Equity	1,03	0,0004	0,0002	0,0002	1,95%	1,35%	0,1
NFLX US Equity	0,94	0,0014	0,0002	0,0013	3,79%	3,56%	0,1
AMZN US Equity	1,22	0,0007	0,0003	0,0004	2,67%	2,08%	0,1
QCOM US Equity	1,04	0,0004	0,0002	0,0002	1,96%	1,35%	0,1
AMD US Equity	1,41	0,0012	0,0004	0,0009	3,52%	2,93%	0,1
INTC US Equity	1,02	0,0004	0,0002	0,0002	1,87%	1,24%	0,1
Nasdaq	1,00	0,0002					
Portfolio	**1,06**	**0,0003**	**0,0002**	**0,00004**	**1,59%**	**0,64%**	1

Daily portfolio σ	1,46%	1,46%	TRUE
1d VaR @ 95%	**23 993**		

VaR – Factor model

- **If a one-factor model is not enough, the precision can be improved by using multiple (*k*) factors:**

$$R_i = \alpha_i + \beta_{i1} f_1 + \cdots + \beta_{iK} f_K + \epsilon_i$$

- In this case, equation $\Sigma = \beta\beta'\sigma_m^2 + D_\epsilon$ becomes:

$$\Sigma = \beta_1 \beta_1' \sigma_1^2 + \cdots + \beta_K \beta_K' \sigma_K^2 + D_\epsilon$$

$$\Downarrow$$

$$V(R_p) \rightarrow (w'\beta\beta'w)\,\sigma_m^2 = (\beta_p \sigma_m)^2$$

$$\Updownarrow$$

$$V(R_p) \rightarrow (\beta_{1p}\sigma_1)^2 + \ldots + (\beta_{Kp}\sigma_K)^2$$

VaR – Factor model

- **One of the key questions is how to choose the risk factors.**
- A common methodology is to use factors that are expected to be relevant to explain asset returns, according to the literature and market practices, e.g. risk-free short term interest rates or measures for the slope of the yield curve.
- **Factors in the Fama-French (1993) model:***

(i) Difference between the market return and the risk-free rate (CAPM)
(ii) Small minus big capitalization
(iii) High minus low book-to-market ratio

Factor	Loading
β_1	1,14
β_2	0,14
β_3	-0,52

Factor Σ	β1	β2	β3
β1	0,0002	0,0000	0,0000
β2	0,0000	0,0000	0,0000
β3	0,0000	0,0000	0,0000

	Portfolio
Variance	0,000187
Stdev	1,37%
1d VaR @ 95%	**22 490**

*Fama, E. F. and French, K. R. (1993), "Common risk factors in the returns on stocks and bonds“, Journal of Financial Economics.

VaR – Full model

- **The portfolio VaR can be obtained straight from the distribution of the returns of the portfolio.**
- If the individual asset returns follow a normal distribution, then the portfolio returns also follows normal distribution.
- VaR of a portfolio of stocks is measured by

$$VaR = -z_{\alpha}\sigma_{p}V = -z_{\alpha}V\sqrt{w'\Sigma w}$$

where Σ is the variance-covariance matrix and w the weight in each stock

VaR – Full model

	Daily σ^2_i	w	**1d VaR @ 95%**
MSFT US Equity	0,0003	0,1	2 738
AAPL US Equity	0,0005	0,1	3 748
YHOO US Equity	0,0006	0,1	4 159
ORCL US Equity	0,0003	0,1	3 043
CSCO US Equity	0,0004	0,1	3 213
NFLX US Equity	0,0014	0,1	6 228
AMZN US Equity	0,0007	0,1	4 386
QCOM US Equity	0,0004	0,1	3 231
AMD US Equity	0,0012	0,1	5 782
INTC US Equity	0,0004	0,1	3 081
Portfolio	0,025%		
Daily portfolio σ	1,58%		
Diversfied 1d VaR @ 95%			**25 958**
Undiversified 1d VaR @ 95%			**39 610**

$$\text{Portfolio VAR} = \text{VAR}_p = \alpha\sigma_p W = \alpha\sqrt{x'\Sigma x}$$

VaR for Stocks - Conclusions

Notional (in Usd)	1 000 000	
Confidence Level	95%	-1,64
Horizon (in days)	1	

	Daily σ_P	**1d VaR @ 95%**
Undiversified		39 610
Full model	1,58%	25 958
Diagonal model	1,59%	26 207
Beta model	1,46%	23 993
Factor model	1,37%	22 490

- VaR is always given by the Portfolio Value x N() x σ_p , being the latter the only parameter changing.
- The undiversified model provides the highest VaR, as it doesn't capture the diversification effect.
- The VaR with the diagonal model is the 2nd highest, as it considers the specific and the systematic risk.
- The beta and factor models provides lower VaR values, as they ignore the specific risk.

4.6. VaR for bond portfolios

VaR for Bonds

- Theoretically we could use the same methodology as before => a bond VaR would be given by $VaR = -VZ_{\alpha}\,\sigma.$
- However, the volatility estimation for bond prices presents some difficulties:
  - Bonds converge to par (pull-to-par)
  - Maturity changes along time
- The risk profile of bonds change when they get closer to maturity.

VaR for Bonds

- It is not possible to use time-series of bond prices to calculate the VaR of a bond portfolio.
- Bonds have to be mapped on yields according to their current profile (e.g. maturity).
- Mapping the exposures to risk factors is the only solution when the characteristics of assets change over time.
- Mapping – process by which the values of the portfolio positions are replaced by exposures to risk factors.
- Mapping should preserve the market value of the portfolio and ideally its risk.

VaR for Bonds

- Some simplifications are necessary, having in mind that the yield curve can be explained by a limited number of factors.
- The simplest approach is to assume that only parallel shifts in the yield curve occur.
- In this case, only one market variable or factor would have to be known: the size of the parallel shift.
- **The changes in the value of a bond portfolio can then be calculated using the modified duration relationship:** $(dP/P) = -D^* \times (dy)$
- **Volatility of bond prices:** $\sigma(dP/P) = |D^*| \times \sigma(dy)$
- **VaR of bond prices:** $\mathrm{VAR}(dP/P) = |D^*| \times \mathrm{VAR}(dy)$

VaR for Bonds

- In practice, the risk structure is often simplified to a single factor.

- Duration model – assumes that the yield curve only faces parallel movements (upward or downward).

- The volatility of yield changes is the same for all maturities, …
- … even though the volatility of bond prices differs according to the modified duration.

- Problem: Does this hold in reality?

VaR for Bonds

- With only parallel movements in the yield curve, the Yield VaR (last column) should be equal for all.
- According to the following table, they are actually similar, even though not equal, as longer maturities exhibit lower yields.

Risk of U.S. Bonds (Monthly VAR at 95 Percent Level)

Term (year)	Returns VAR (%)	Yield (%)	Modified Duration	Yield VAR (%)
1	0.470	5.83	0.945	0.497
2	0.987	5.71	1.892	0.522
3	1.484	5.81	2.835	0.523
4	1.971	5.89	3.777	0.522
5	2.426	5.96	4.719	0.514
7	3.192	6.07	6.599	0.484
9	3.913	6.20	8.475	0.462
10	4.250	6.26	9.411	0.452
15	6.234	6.59	14.072	0.443
20	8.146	6.74	18.737	0.435
30	11.119	6.72	28.111	0.396

Source: Jorion (2007)

VaR for Bonds

- The same pattern can be found in correlations.
- Actually, the yields exhibit high correlations, namely for close maturities, as for more distant maturities correlations decrease.
- Conclusion: more than 1 factor seems to be necessary to explain the yield curve shifts and therefore to calculate the VaR of a bond portfolio.

Correlation Matrix of U.S. Bonds

Term (year)	1Y	2Y	3Y	4Y	5Y	7Y	9Y	10Y	15Y	20Y	30Y
1	1										
2	0.897	1									
3	0.886	0.991	1								
4	0.866	0.976	0.994	1							
5	0.855	0.966	0.988	0.998	1						
7	0.825	0.936	0.965	0.982	0.990	1					
9	0.796	0.909	0.942	0.964	0.975	0.996	1				
10	0.788	0.903	0.937	0.959	0.971	0.994	0.999	1			
15	0.740	0.853	0.891	0.915	0.930	0.961	0.976	0.981	1		
20	0.679	0.791	0.832	0.860	0.878	0.919	0.942	0.951	0.991	1	
30	0.644	0.761	0.801	0.831	0.853	0.902	0.931	0.943	0.975	0.986	1

Source: Jorion (2007)

Mapping

- Interest rate risk in bonds can be measured by a mapping system, using different factors that have to be identified (or assumed):

  - The risk of a bond is analyzed such that a bond is a portfolio of zero coupon instruments
  - Volatility is computed from the combination of the risk of the several zero-coupons

- **Mapping approaches:**

  (i) Cash-flow mapping - bond risk is decomposed into the present value of each bond cash flow, that corresponds to the cash-flows of zero-coupons, being these cash-flows grouped into maturity buckets.

  (ii) Maturity (principal) mapping - bond risk associated with bond maturity.

  (iii) Duration mapping - bond risk is associated with zero coupon bond with equal duration.

Mapping

- Choose as market variables the prices of zero-coupon bonds with standard maturities: 1m, 3m, 6m, 1y, 2y, 5y, 7y, 10y and 30y.
- To calculate VaR, the cash-flows from instruments in the portfolio are mapped into cash-flows occurring on the standard maturity dates.
- The relevant Σ matrix is estimated from the zero-coupon bond returns.
- **Example:** $1M position in a Treasury bond with:

- term to maturity = 1.2 years
- coupon – 6% semiannually => coupons are paid in 0.2, 0.7 and 1.2 years (2.4, 8.4 and 14.4 months, respectively), while the principal is paid in 1.2 years too.
- This bond is seen as: $30,000 position in 0.2-year zero-coupon bond + $30,000 position in a 0.7-year zero-coupon bond + $1.03M position in a 1.2-year zero-coupon bond.

Mapping

- The position in the 0.2-year bond is then replaced by an approximately equivalent position in 1-month and 3-month zero-coupon bonds;
- The position in the 0.7-year bond is replaced by an approximately equivalent position in 6-month and 1-year zero-coupon bonds; and
- The position in the 1.2-year bond is replaced by an approximately equivalent position in 1-year and 2-year zero-coupon bonds.

$\Downarrow$

- The position in the 1.2-year coupon-bearing bond is regarded as a position in zero-coupon bonds with maturities of 1m, 3m, 6m, 1y and 2y.

ZC returns

- When measuring interest rate risk, zero-coupons are risk factors that represent different maturities:
- The price of the zero-coupon bond with simple compounding is:

$$P_{ZC,T} = \frac{100}{(1+y)^T},$$ where y is the relevant spot rate

- With continuous compounding, one gets: $P_{ZC,T} = 100 \times e^{-yT}$
- The return of the zero-coupon bond, measured as the holding yield between *t* and *t+1* is:

$$R_{t+1} = ln\left({}^{P_{t+1}}/_{P_t}\right)$$

Single coupon-paying bond example

- Compute the monthly VaR @ 95% for a 100,000€ notional investment in a 2y bond with a 2% annual coupon.

Spot Rate	DF
0,09%	0,9991
0,25%	0,9951

$$P_{2y,2\%} = 2\% \cdot DF_1 + 102\% \cdot DF_2$$

$$DF_t = \left(\frac{1}{1+s_t}\right)^t, s_t \textit{ is the spot rate}$$

With discrete compounding

- The 2y coupon bond can be decomposed into 2 different bonds:
  - 1y bond that pays 2% at maturity
  - 2y bond that pays 102% at maturity

Single bond example – Cash-flow Mapping

CF mapping

- **Present values:**
  - **1st cash-flow = 2 x 0,9991 = 1,998**
  - **2nd cash-flow = 102 x 0,9951 = 101,495**
- **Bond Price:**

  - Sum of NPF (cash-flows) = 1,998 + 101,495 = 103,49

- **Weights:**
  - **1st cash-flow = 1,998 / Price = 1,998 / 103,49 = 1,93%**
  - **2nd cash-flow = 101,495 / Price = 101,495 / 103,49 =98,07%**

Single bond example – Cash-flow Mapping

- The monthly variance of the portfolio composed by the 2 cash-flows is calculated from $\sigma_p^2 = w'\Sigma w$, considering the variances and covariances of the 1y and 2y interest rates and the weights calculated in the previous slide:

w
1,93%
98,07%

- The variance-covariance matrix is given by:

$\Sigma_{monthly}$	1	2
1	0,0000016	0,0000041
2	0,0000041	0,0000122

- $\sigma_p^2 = w'\Sigma w$ = 0,0012% => Monthly or 20-day VaR @ 95% is just $VaR = -VZ_\alpha \sigma = -103493{,}53€ \cdot (-1{,}64) \cdot 0{,}34\% = 586{,}37$

Single bond example – Maturity Mapping

- The bond has 2y maturity and the risk of the 2y ZC is:

$$\sigma_{Bond} = \sigma_{2y\ ZC} = 0{,}35\%$$

- Monthly VaR @ 95% is just:

$$20d\ VaR\ @\ 95\% = -103493{,}53€ \cdot (-1{,}64) \cdot 0{,}35\% = 594{,}02$$

	Maturity	Notional	Coupon	Price	Market value	$\sigma_{monthly}$	σ^$2_{monthly}$	Monthly VaR @ 95%
2y Bond	2	100 000	2,00%	103,49	103493,5297	0,35%	0,0012%	594,02

T	Spot rate	DF	CFBond1	PV
0			-103,49	
1	0,09%	0,9991	2	2,00
2	0,25%	0,9951	102	101,50

$\Sigma_{monthly}$	1	2
1	0,0000016	0,0000041
2	0,0000041	0,0000122

- This VaR is higher than in cash-flow mapping, as in the latter the volatility is lower, due to the lower volatility of the 1y vis-à-vis the 2y interest rate (as in maturity mapping only the maturity of the residual cash-flow is taken into account).

Single bond example – Duration Mapping

- The bond has MD=1,98 (D is similar to MD, as interest rates are very low).
- As MD is between 2 vertices (1y and 2y interest rates), the portfolio volatility (in this case the portfolio corresponds to the 2 cash-flows paid by the asset) may be calculated as a linear combination of the volatilities of interest rates in the 2 adjacent vertices (1y and 2y): $\sigma_{Bond} = \frac{\sigma_{1y\,ZC}}{(2-1)} + \frac{(1{,}98-1)\cdot(\sigma_{2y\,ZC}-\sigma_{1y\,ZC})}{(2-1)} = 0{,}34\%$

	Maturity	Notional	Coupon	Price	D	MD	Market value	$\sigma^2_{monthly}$	Monthly VaR @ 95%
2y Bond	2	100 000	2,00%	103,49	1,98	1,98	103 493,53	0,0012%	586,75

T	Spot rate	DF	CFBond1	PV	PV X t
0			-103,49		
1	0,09%	0,9991	2	2,00	2,00
2	0,25%	0,9951	102	101,50	202,99

- Obviously, the exercise can be done by using MD – in that case it would be MD mapping, even though in this example the results would be similar, as interest rates are very low.

Portfolio of Bonds

- Compute the monthly VaR @ 95% for a portfolio of bonds = previous investment + 250,000€ notional investment in a 4y bond with 1% coupon:

	Maturity	Notional	Coupon	Price	D	MD	Market value
2y Bond	2	100000	0,02	103,49	1,98	1,98	103 493,53
4y Bond	4	250000	0,01	99,77	3,94	3,90	249 413,82
Portfolio	**3,41**	**350000**			**3,37**	**3,33**	**352 907,35**

Portfolio of Bonds– Cash-flow mapping

- Calculated just like for a single asset, from the corresponding weights for all cash-flows and their variance-covariance matrix:

T	Spot rate	DF	CF_1	CF_2	Total CF	PV	**W Portfolio**
0	0,00%	0					
1	0,09%	0,999059	2000	2500	4500	4 495,77	**1%**
2	0,25%	0,995053	102000	2500	104500	103 983,04	**29%**
3	0,49%	0,985386		2500	2500	2 463,46	**1%**
4	1,07%	0,958278		252500	252500	241 965,08	**69%**
						352 907,35	**100%**

$\Sigma_{monthly}$	1	2	3	4
1	0,000002	0,000004	0,000006	0,000008
2	0,000004	0,000012	0,000019	0,000025
3	0,000006	0,000019	0,000032	0,000042
4	0,000008	0,000025	0,000042	0,000057

$$Monthly\ \sigma_p = \sqrt{w' \cdot \Sigma \cdot w} = 0{,}62\%$$

Higher than with the single asset (0,35%), as the 2nd asset is more volatile, due to the higher volatility of the 4y interest rate and the higher weight of that cash-flow in the portfolio (around 70%).

Portfolio of Bonds– Cash-flow mapping

1-month portfolio VaR:

$$20d\ VaR\ @\ 95\% = -V \cdot z_{0,05} \cdot Monthly\ \sigma_p$$
$$20d\ VaR\ @\ 95\% = -352907{,}35€ \cdot (-1{,}64) \cdot 0{,}62\%$$

Diversified vs Undiversified VaR (as the sum of the individual VaR of both assets):

	$\sigma_{monthly}$	**20d VaR @ 95%**
2y Bond	0,34%	586,37
4y Bond	0,74%	3039,56
Portfolio	**0,62%**	3594,63

Diversified 20d VaR @ 95%	3 594,63
Undiversified 20d VaR @ 95%	3 625,92

Portfolio of Bonds– Maturity mapping

- Being the residual maturity of the portfolio = 3,41y, the volatility of the portfolio may be calculated by linear interpolation from the volatility of the nearest vertices (3y and 4y): $$\sigma_{Portfolio} = \frac{\sigma_{3y\ ZC}}{(4-3)}(3{,}41-3)\cdot\frac{(\sigma_{4y\ ZC}-\sigma_{3y\ ZC})}{(4-3)} = 0{,}64\%$$

- Monthly VaR @ 95% is just:

$$20d\ VaR\ @\ 95\% = -352907{,}35€ \cdot (-1{,}64) \cdot 0{,}64\% = 3726{,}37$$

Higher than the VaR with the Cash-flow mapping, as it provided a higher weight to the 4y yield, which is more volatile.

Portfolio of Bonds– Duration mapping

- In the same vein, as the portfolio duration = 3,37, the volatility of the portfolio may be calculated by linear interpolation from the volatility of the nearest vertices (3y and 4y):

$$\sigma_{Portfolio} = \frac{\sigma_{3y\ ZC}}{(4-3)} + (3{,}37-3) \cdot \frac{(\sigma_{4y\ ZC} - \sigma_{3y\ ZC})}{(4-3)} = 0{,}63\%$$

- Monthly VaR @ 95%:

$$20d\ VaR\ @\ 95\% = -352907{,}35€ \cdot (-1{,}64) \cdot 0{,}63\% = 3673{,}33$$

Portfolio of Bonds: Duration mapping general approach

- An alternative approach to linear interpolations is duration matching.
- Assuming that the duration (or the MD) is between 2 knots D_1 and D_2 and x is the weight of the first knot, the portfolio duration D_P will be matched if

$$xD_1 + (1 - x)D_2 = D_p$$

or

$$x = (D_2 - D_p)/(D_2 - D_1)$$

- However, this approach may not create a portfolio with the same risk as the original portfolio.

$$\Downarrow$$

- Variance matching:

$$V(R_p) = x^2\sigma_1^2 + (1 - x)^2\sigma_2^2 + 2x(1 - x)\rho\sigma_1\sigma_2$$

$$\Downarrow$$

Portfolio of Bonds: Duration mapping general approach

- From the previous equation, $V(R_p) = x^2\sigma_1^2 + (1-x)^2\sigma_2^2 + 2x(1-x)\rho\sigma_1\sigma_2$
- *x* will be the solution to the following:

$$(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)x^2 + 2(-\sigma_2^2 + \rho\sigma_1\sigma_2)x + (\sigma_2^2 - \sigma_p^2) = 0$$

- Being a second order equation, it will provide 2 solutions and the one to be chosen must be that between 0 and 1.

Conclusions

- ZC's are the relevant risk factors for estimating IR risk.
- 3 methods to estimate volatility:
  - Cash-flow mapping
  - Maturity mapping
  - Duration mapping
- Knowing your exposure to the different risk factors is key for hedging IR risk.
- For very large portfolios, with exposures to cash-flows being paid in many different maturities, one may assume that the yields in several maturities are almost perfectly correlated and their volatilities are similar, in order to avoid too much information on the yield curve.

Bucketing

- Previously, each cash flow was set in each standard maturity for which a discount factor, standard deviation and correlation were available.
- But in practice, a bond portfolio will comprise a large number of payment dates and it is impossible to estimate such a huge number of parameters.
- Therefore, one may consider a set of maturities (buckets) in which the cash flows will be allocated such that the bucket exposure replicates the original investment risk.
- The risk buckets usually include 1M, 3M, 6M, 1Y, 2Y, 3Y, 4Y, 5Y, 6Y, 7Y, 8Y, 9Y, 10Y, 15Y, 20Y, 25Y, 30Y.

Bucketing

- Criteria for the allocation of cash flow to set the buckets:

  - Cash-flow: bucket allocation according to the **duration or maturity of the several cash-flows** (as we're dealing with cash-flows that correspond to zero-coupon bonds, the maturity and the duration are the same).
  - Duration: bucket allocation according to the **duration or maturity of risk factors**.

- This allocation must be done for each cash-flow in the portfolio.

- Finally, the sum of the total cash flow allocated to each bucket is computed to obtain the distribution of the portfolio cash flows by the buckets that better replicate the original portfolio.

Bucketing

- 20d VaR @ 95% for the following portfolio (both bonds with annual coupons):
- a

	Maturity	Coupon	Price	Mac Duration	Notional	Market value
$Bond_1$	2,25	2,00%	105,29	2,19	100 000	105 290
$Bond_2$	4,6	1,00%	100,65	4,50	250 000	251 623
Portfolio	3,91			3,82	350 000	356 914

Cash-flow Mapping

- We will have to do the stripping of the coupons of both bonds:

T	Spot rate	DF	CF_1	CF_2	Total CF	PV	w
0,25	0,08%	0,9998	2 000	0	2 000	2 000	0,56%
0,6	0,07%	0,9996	0	2 500	2 500	2 499	0,70%
1,25	0,13%	0,9983	2 000	0	2 000	1 997	0,56%
1,6	0,19%	0,9970	0	2 500	2 500	2 493	0,70%
2,25	0,31%	0,9931	102 000	0	102 000	101 294	28,38%
2,6	0,39%	0,9898	0	2 500	2 500	2 475	0,69%
3,25	0,56%	0,9819	0	0	0	0	0,00%
3,6	0,66%	0,9765	0	2 500	2 500	2 441	0,68%
4,25	0,85%	0,9647	0	0	0	0	0,00%
4,6	0,95%	0,9573	0	252 500	252 500	241 716	67,72%
Total			**106 000**	**262 500**	**368 500**	**356 914**	

Cash-flow Mapping

- To calculate σ_p and the VaR, the bucketing approach involves using the VCV matrix only for the risk factors aggregated by maturity or duration.

- To estimate σ_p we need:
  - To estimate spot rates for *non-standard* maturities (done by linear interpolations of spot rates for standard maturities which are assumed as given)
  - To estimate a 10x10 Σ matrix (as we have 10 different maturities considering all bond cash-flows)

- The problem increases when we add bonds to our portfolio.

Cash-flow Mapping

Var-CoVar Matrix for the daily returns of zero-coupons:

Σ_{daily}	0,25	0,6	1,25	1,6	2,25	2,6	3,25	3,6	4,25	4,6
0,25	0,000000006	0,000000006	0,000000009	0,000000011	0,000000013	0,000000014	0,000000014	0,000000014	0,000000014	0,000000015
0,6	0,000000006	0,000000018	0,000000041	0,000000052	0,000000072	0,000000082	0,000000099	0,000000107	0,000000120	0,000000126
1,25	0,000000009	0,000000041	0,000000146	0,000000206	0,000000322	0,000000373	0,000000462	0,000000503	0,000000570	0,000000601
1,6	0,000000011	0,000000052	0,000000206	0,000000298	0,000000479	0,000000559	0,000000700	0,000000763	0,000000869	0,000000918
2,25	0,000000013	0,000000072	0,000000322	0,000000479	0,000000793	0,000000936	0,000001190	0,000001304	0,000001498	0,000001587
2,6	0,000000014	0,000000082	0,000000373	0,000000559	0,000000936	0,000001113	0,000001427	0,000001569	0,000001813	0,000001925
3,25	0,000000014	0,000000099	0,000000462	0,000000700	0,000001190	0,000001427	0,000001856	0,000002054	0,000002397	0,000002556
3,6	0,000000014	0,000000107	0,000000503	0,000000763	0,000001304	0,000001569	0,000002054	0,000002280	0,000002675	0,000002859
4,25	0,000000014	0,000000120	0,000000570	0,000000869	0,000001498	0,000001813	0,000002397	0,000002675	0,000003167	0,000003400
4,6	0,000000015	0,000000126	0,000000601	0,000000918	0,000001587	0,000001925	0,000002556	0,000002859	0,000003400	0,000003660

Cash-flow Mapping

Var-CoVar Matrix for the monthly returns of zero-coupons:

$\Sigma_{monthly}$	0,25	0,6	1,25	1,6	2,25	2,6	3,25	3,6	4,25	4,6
0,25	0,0000001	0,0000001	0,0000002	0,0000002	0,0000003	0,0000003	0,0000003	0,0000003	0,0000003	0,0000003
0,6	0,0000001	0,0000004	0,0000008	0,0000010	0,0000014	0,0000016	0,0000020	0,0000021	0,0000024	0,0000025
1,25	0,0000002	0,0000008	0,0000029	0,0000041	0,0000064	0,0000075	0,0000092	0,0000101	0,0000114	0,0000120
1,6	0,0000002	0,0000010	0,0000041	0,0000060	0,0000096	0,0000112	0,0000140	0,0000153	0,0000174	0,0000184
2,25	0,0000003	0,0000014	0,0000064	0,0000096	0,0000159	0,0000187	0,0000238	0,0000261	0,0000300	0,0000317
2,6	0,0000003	0,0000016	0,0000075	0,0000112	0,0000187	0,0000223	0,0000285	0,0000314	0,0000363	0,0000385
3,25	0,0000003	0,0000020	0,0000092	0,0000140	0,0000238	0,0000285	0,0000371	0,0000411	0,0000479	0,0000511
3,6	0,0000003	0,0000021	0,0000101	0,0000153	0,0000261	0,0000314	0,0000411	0,0000456	0,0000535	0,0000572
4,25	0,0000003	0,0000024	0,0000114	0,0000174	0,0000300	0,0000363	0,0000479	0,0000535	0,0000633	0,0000680
4,6	0,0000003	0,0000025	0,0000120	0,0000184	0,0000317	0,0000385	0,0000511	0,0000572	0,0000680	0,0000732

$$Monthly\ \sigma_p = \sqrt{w' \cdot \Sigma \cdot w} = 0{,}6963\%$$

$$20d\ VaR\ @\ 95\% = -V \cdot z_{0,05} \cdot Monthly\ \sigma_p$$

$$= \text{-}356.914 \cdot (\text{-}1{,}65) \cdot 0{,}6963\% =$$

$$= 4087{,}56$$

Duration Bucketing

- Each cash flow must be allocated between the standard buckets, but this time considering duration (or maturity) buckets and their weights.
- The number of maturities will now be lower, as we are working with a fixed set of maturities, regardless the maturities of the different cash-flows.
- Assuming that the duration of the allocation is the same as the original one:

$$w_L D_L + w_H D_H = D_{cf}$$

where
D_L= duration of the bucket with the lower maturity
D_H= duration of the bucket with the higher maturity
D_H= duration of the cash-flow to allocate
w_L= weight of the bucket with the lower maturity
w_H= weight of the bucket with the lower maturity (1-w_L)

Duration Bucketing

Example

CF1 2000€ in 1.25y

$$w_L D_L + (1 - w_{\boldsymbol{L}}) D_H = D_{cf}$$

$$w_L \cdot 1 + (1 - w_{\boldsymbol{L}}) \cdot 2 = 1.25$$

$$w_{bucket\ 1y} = 75\%$$

$$w_{bucket\ 2y} = 25\%$$

$$w_{bucket\ 2y} = (D_{2.25} - D_3)/(D_2 - D_3)$$

Generalization

$$w_L D_L + (1 - w_{\boldsymbol{L}}) D_H = D_{cf}$$

$$\Leftrightarrow w_L D_L + D_H - w_{\boldsymbol{L}} D_H = D_{cf}$$

$$\Leftrightarrow w_L D_L - w_{\boldsymbol{L}} D_H = D_{cf} - D_H$$

$$\Leftrightarrow w_L (D_L - D_H) = D_{cf} - D_H$$

$$\Leftrightarrow w_L = (D_{cf} - D_H)/(D_L - D_H)$$

CF1 102000€ in 2.25y

$$w_{bucket\ 2y} = (2.25 - 3)/\,(2 - 3) = 75\%$$

$$w_{bucket\ 3y} = 1 - w_{bucket\ 2y} = 25\%$$

Duration Bucketing

- Calculation of buckets' weights:

T	Spot rate	DF	$CFBond_1$	PV	PV X t	Bucket	w_L	w_H	CF1
0									
0,25	0,08%	0,9998	2,00	2,00		0,25	1	0	2000
1,25	0,13%	0,9983	2,00	2,00	2,50	1,00	0,75	0,25	2000
2,25	0,31%	0,9931	102,00	101,29	227,91	2,00	0,75	0,25	102000
3,25	0,56%	0,9819		0,00	0,00	3,00	0,75	0,25	0
4,25	0,85%	0,9647		0,00	0,00	4,00	0,75	0,25	0
				105,29		5,00			106000

T	Spot rate	DF	$CFBond_2$	PV	PV X t	Bucket	w_L	w_H	CF2
0						0,25	0	0	0
0,6	0,08%	0,9995	1	1,00	0,60	0,5	0,8	0,2	2500
1,6	0,19%	0,9970	1	1,00	1,60	1	0,4	0,6	2500
2,6	0,39%	0,9898	1	0,99	2,57	2	0,4	0,6	2500
3,6	0,66%	0,9765	1	0,98	3,52	3	0,4	0,6	2500
4,6	0,95%	0,9573	101	96,69	444,76	4	0,4	0,6	252500
				100,65		5			262500

Duration Bucketing

- Calculation of buckets' amounts for Bond 1:

Bucket	Spot rate	DF	CF_1
0,25	0,08%	0,9998	2000
0,50	0,07%	0,9996	0
1,00	0,09%	0,9991	1 500
2,00	0,25%	0,9951	77 000
3,00	0,49%	0,9854	25 500
4,00	0,78%	0,9695	0
5,00	1,07%	0,9481	0
Total			106 000

$\rightarrow \quad 100\% \cdot 2000 = 2000$

$\rightarrow \quad 75\% \cdot 2000 = 1500$

$\rightarrow \quad 25\% \cdot 2000 + 75\% \cdot 102000 = 77000$

$\rightarrow \quad 25\% \cdot 102000 = 25500$

Duration Bucketing

- Calculation of buckets' amounts for the total portfolio:

Bucket	Spot rate	DF	CF_1	CF_2	Total CF	PV	w
0,25	0,08%	0,9998	2000		2 000	2 000	0,56%
0,50	0,07%	0,9996	0	2 000	2 000	1 999	0,56%
1,00	0,09%	0,9991	1 500	1 500	3 000	2 997	0,84%
2,00	0,25%	0,9951	77 000	2 500	79 500	79 107	22,18%
3,00	0,49%	0,9854	25 500	2 500	28 000	27 591	7,73%
4,00	0,78%	0,9695	0	102 500	102 500	99 376	27,86%
5,00	1,07%	0,9481	0	151 500	151 500	143 640	40,27%
Total			106 000	262 500		356 710	100,00%

weights to be used in the VaR computation

Duration Bucketing

- To estimate σ_p we need to estimate the Var-Covar matrix for the relevant risk (duration/maturity) buckets:

$\Sigma_{monthly}$	0,25	0,5	1	2	3	4	5
0,25	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000	0,000000
0,50	0,000000	0,000000	0,000001	0,000001	0,000001	0,000002	0,000002
1,00	0,000000	0,000001	0,000002	0,000004	0,000006	0,000008	0,000009
2,00	0,000000	0,000001	0,000004	0,000012	0,000019	0,000025	0,000029
3,00	0,000000	0,000001	0,000006	0,000019	0,000032	0,000042	0,000050
4,00	0,000000	0,000002	0,000008	0,000025	0,000042	0,000057	0,000069
5,00	0,000000	0,000002	0,000009	0,000029	0,000050	0,000069	0,000086

Duration Bucketing

- From the weights and the Var-Covar matrix for the risk factors, we may calculate the monthly VaR:

$$\text{Monthly } \sigma_p = \sqrt{w' \cdot \Sigma \cdot w} = 0{,}70\%$$

$$20d\ VaR\ @\ 95\% = -V \cdot z_{0{,}05} \cdot \text{Monthly } \sigma_p$$
$$= \text{-}356.913 \cdot (\text{-}1{,}65) \cdot 0{,}6962\% = 4087{,}33$$

- Conclusion: the VaR obtained from duration bucketing is very similar to the one with cash-flow bucketing, as the slope of the yield curve is very low => the interest rates used to discount cash-flows according to their effective maturity and to the maturities chosen as risk factors are very similar.

FRN

- The price of a FRN is given by the sum of the present value of:
  - Next coupon (C, set at the previous coupon payment date)
  - Bond price in the next coupon payment date (time t), which will be the redemption value.

- The VaR for a FRN can be calculated as for a fixed rate bond with:
  - Maturity in the next coupon date
  - Duration equal to maturity

FRN

Example:

- Nominal value of the exposure = 10 M€.
- Coupon rate = 2.5% (yearly)
- Coupon payments – each semester
- Next coupon payment – in 4 months
- Current 4-month spot rate = 3%.
- Price: $P = \frac{100 \times 0.025/2 + 100}{1 + 0.03 \times 4/12} = 100.25$
- Market value of Portfolio = €10,024,752
- Maturity and duration of the portfolio = 0.33(3) years

FRN

- Market value of Portfolio = €10,024,752
- Maturity and duration of the portfolio = 0.33(3) years
- As there is no available information for variances and covariances of 0.33(3) years, we need to allocate cash flows to the 3m and 6m buckets.
- Using the duration bucketing criterium:

$$w_L D_L + (1 - w_L) D_H = D_{cf}$$

$$w_L \cdot 0.25 + (1 - w_L) \cdot 0.5 = 0.33$$

$$w_{bucket\ 0.25y} = 67\%$$

$$w_{bucket\ 0.5y} = 33\%$$

FRN

Key data:

Bucket	Spot	CF	**w**
0,25	0,08%	6 683 168,32	**66,6667%**
0,50	0,07%	3 341 584,16	**33,3333%**
Total		10 024 752,48	100%

Variance-Covariance Matrix of the daily returns of the risk factors (3m and 6m):

Bucket	0,25	0,50
0,25	0,00000012	0,00000010
0,50	0,00000010	0,00000025

FRN

- FRN risk is just:

$$\text{Monthly } \sigma_p = \sqrt{w' \cdot \Sigma \cdot w} = 0{,}04\%$$

$$20d\ VaR\ @\ 95\% = -V \cdot z_{0,05} \cdot Monthly\ \sigma_p = -\ 10\ 024\ 752{,}48 \cdot (-\ 1{,}65) \cdot 0{,}04\% = 5\ 873$$

4.7. Backtesting VaR

Stressed VaR and Backtests

- Additionally to the VaR and calculation, stressed VaR and ES, as well as backtests, are usually performed.

- Stressed VaR and ES - done by assuming extreme values for the volatilities and correlations, e.g. those observed in previous financial crisis.

- Backtests - comparison between losses observed in the past and losses estimated by the VaR, to determine whether the % of days with losses > VaR exceeded the VaR confidence level.