Waiting-Line Exercises Solutions

WL1:

Data:

1/λ = 20 minutes ⇒ λ = 3 clients/hour, Poisson arrivals
1/μ= 12 minutes ⇒ μ = 5 clients/hour, Exponential service
⇒ M/M/1 model

a) $W_q = \frac{\lambda}{\mu\,(\mu-\lambda)} = \frac{3}{5\times 2} = 0.3\ hours = 18\ minutes$

b) $W_s = W_q + 1/\mu = 18 + 12 =$ *30 minutes*

c) $L_q = \frac{\lambda^2}{\mu\,(\mu-\lambda)} = \frac{9}{5(5-3)} = 0.9\ clients, or$ $L_q = \lambda \times W_q = 3 \times (18/60) =$ *0.9 clients*

d) $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{3}{2} = 1.5\ clients$

e) Utilization rate = ρ= λ/ μ = 3/5= 0.6, or *60%*

f) Probability of *2* elements in the system

P_2= P(n>1)-P(n>2) = $\left(\frac{3}{5}\right)^2 - \left(\frac{3}{5}\right)^3 = 0.1440$

WL_2:

Data:

λ = 25 clients/hour, Poisson arrivals
$1/\mu$ = 2 minutes $\Rightarrow \mu$ = 60/2 = 30 clients/hour, Exponential service
$\Rightarrow$ M/M/1 model

a) P0 = 1-ρ = 1-25/30 = *0.1667*

b) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{25^2}{30\times(30-25)} = 4.167\ clients$

c) From Little's Law, we can deduce: Wq = Lq/λ = 4.167/25 = 0.1667 hours = x 60 = ***10 minutes***

Ws = 10 + 2 = *12 minutes*

d) λ = 25 clients/hour

μ = 3600/90 = 40 clients/hour

M/M/1 model

$$\mathrm{Wq} = \frac{25}{40(40-25)} = 0.0417 \text{ hours} = 2.5 \text{ min}$$

Ws = 2.5+1.5 = *4 minutes*

The time was reduced by 8 minutes.

e) λ = 25 clients/hour

μ = 30 clients/hour

In this case, there are 2 independent servers, and thus the model is **M/M/2**.

$$P_0 = \frac{1}{\left(1+\frac{25}{30}\right)+\frac{1}{2}\times\left(\frac{25}{30}\right)^2\times\frac{2\times 30}{2\times 30-25}} = 0.4118$$

$$\mathrm{Wq} = \frac{30\times(\frac{25}{30})^2}{(2-1)!\,(2\times 30-25)^2}\times 0.4118 = 0.007 \text{ hours} = 0.42 \text{ min}$$

Ws = 0.42 minutes + 2 minutes = *2.42 minutes*

Thus, the time was reduced by 9.58 minutes

WL_3:

For the **M/M/1** model, the utilization rate is given by: $\rho = \lambda/\mu$

a) $\rho = 0.5 \Rightarrow 0.5\ \mu = \lambda \Rightarrow L_s = \frac{0{,}5\mu}{(\mu - 0{,}5\mu)} = 1\ client$
b) $\rho = 0.8 \Rightarrow 0.8\ \mu = \lambda \Rightarrow L_s = \frac{0{,}8\mu}{(\mu - 0{,}8\mu)} = 4\ clients$
c) $\rho = 0.9 \Rightarrow 0.9\ \mu = \lambda \Rightarrow L_s = \frac{0{,}9\mu}{(\mu - 0{,}9\mu)} = 9\ clients$
d) $\rho = 0.95 \Rightarrow 0.95\ \mu = \lambda \Rightarrow L_s = \frac{0{,}95\mu}{(\mu - 0{,}95\mu)} = 19\ clients$

WL_4:

Data:

λ = 3 trucks/day, Poisson arrivals

μ = 4 trucks/day (2 men), Exponential service

Although the team is comprised of more than one man, as these work together to unload a truck each time, the model is M/M/1, thus:

$$W_s = \frac{1}{(\mu - \lambda)}$$

$$\text{For example, for } \mu = 4\text{: } W_s = \frac{1}{(4-3)} = 1 \text{ day}$$

Team	μ	Ws
2	4	1 day
3	5	0.5days
4	6	0.3333 days
5	7	0.25 days
6	8	0.20 days

Cost of truck stopped/day	Cost of loading/unloading/day	Total cost/day
$(40 \times 8)\ \times 3 \times 1 = 960$	$2\ \times 12 \times 8 = 192$	1152 e
$(40 \times 8)\ \times 3 \times 0.5 = 480$	$3\ \times 12 \times 8 = 288$	768
$(40 \times 8)\ \times 3 \times 0.3333 =$ 319,97	$4\ \times 12 \times 8 = 384$	**703,97**
$(40 \times 8)\ \times 3 \times 0.25 = 240$	$5\ \times 12 \times 8 = 480$	720

The team should be increased to *4 workers*, as this guarantees the lowest total cost.

WL_5:

Data:

4 men

λ = 1 trucks/day, arrivals according to a Poisson distribution. Exponential service time ⇒ M/M/1 model

Labor cost of the men = 20 Euros/hour/man

Waiting time cost of the trucks = 0.5 Euros/minute = 30 Euros/hour

n	1/μ = 1/n	μ	Wq	Total cost/hour
1	1 hour	1 truck/hour	----	----
2	0.5 hours	2 trucks/hour	0.5 hours	2×20+30×1×0.5 = 55 Euros
3	1/3 hour	3 trucks/hour	1/6 hours	3×20+30×1×(1/6) = 65 Euros
4	¼ hour	4 trucks/hour	1/12 hours	4×20+30×1×(1/12) = 82.5 Euros

$\mathrm{Wq} = \frac{1}{2(2-1)} = 0.5 \text{ hours}; \mathrm{Wq} = \frac{1}{3(3-1)} = 0.166 \text{ hours}; \mathrm{Wq} = \frac{1}{4(4-1)} = 0.0833 \text{ hours}$

The optimum number of men is *2 employees*.

b) Total time = 1/μ1 + 1/μ2 = 30 min + 20 min = 50 minutes

μ = 60/50 = 1.2 trucks/hour
λ = 1 truck/hour

M/M/1 model
Ws = 1/(μ-λ) = 1/(1.2-1)= *5 hours*

WL_6:

Data:

λ = 8 patients/hour

Assuming that the arrivals follow a Poisson distribution and that the service time follows a negative exponential distribution $\Rightarrow$ M/M/1 model

$W_q \leq$ 10 minutes

a) $1/\mu$ = ?

$$W_q = \frac{\lambda}{\mu\,(\mu - \lambda)} \leq \frac{10}{60} \Leftrightarrow \frac{8}{\mu\,(\mu - 8)} \leq \frac{10}{60} \Leftrightarrow \mu^2 - 8\mu - 6 \times 8 \geq 0 \Leftrightarrow \mu = 12 \; or \; \mu = -4$$

Thus, the service time of the team of doctors is 1/12 hour = 5 minutes

W_S = 10 + 5 = 15 minutes (Ws=Waiting time + service time)

$L_s = \lambda \times W_S = 8 \times \left(\frac{15}{60}\right) = 2$ patients

$L_q = W_q \times \lambda$ = (10/60) × 8 = 1.333 patients

$\rho = (\lambda/\mu)$ = 8/12 = 0.667 ➔ 66.67%

$P_0 = 1-\rho$ = 0.333 ➔ 33.33%

b) If W_q > 60 minutes $\Rightarrow$ the patient is transferred

365 days/year (24 hours/day)

5 patients in the queue (6 in the system)

$$P(n > 5) = \left(\frac{\lambda}{\mu}\right)^6 = \left(\frac{8}{12}\right)^6 = 0.087791$$

(8×24h×365) × 0.087791 = 6.152 patients/year

6 patients in the queue (7 in the system, the seventh patient has to be transferred and abandons the system)

$$P(n > 6) = \left(\frac{\lambda}{\mu}\right)^7 = \left(\frac{8}{12}\right)^7 = 0.058528$$

(8×24h × 365) × 0.058528 = 4.102 patients/year

WL_7:

Data:

Massage service:

λ = 4 passengers /hour, arrivals follow a Poisson distribution

1/ μ = 12 minutes ⇒ μ = 60/12 = 5 passengers /hour

M/D/1

Beauty service

λ = 2 passengers /hour, arrivals follow a Poisson distribution

1/ μ = 20 minutes ⇒ μ = 60/20 = 3 passengers /hour

Service time follows a negative exponential distribution

M/M/1

a)

$$W_q = \frac{4}{2 \times \mu\ (\mu - \lambda)} = \frac{4}{2 \times 5\ (5-4)} = 0{,}4\ hours$$

b) $P(n > k) = \rho^{k+1}$

$P(n \geq 3) = P(n>2) = (2/3)^{2+1} = 0.2963$

c) $P\ (n > 3) - P(n > 4) = (2/3)^{3+1} - (2/3)^{4+1} = 0.0658$

d) $Ls = Ls_M + Ls_B$

Beauty service

$$L_{sB} = \frac{\lambda}{(\mu - \lambda)} = \frac{2}{(3-2)} = 2\ passangers$$

Massage service:

$$W_q = 0{,}4\ hours \quad \Rightarrow \quad L_q = \lambda \times W_q = 4 \times 0.4 = 1.6 \text{ passangers} \quad \Rightarrow \quad L_S = L_S + \left(\frac{\lambda}{\mu}\right) = 1.6 + \left(\frac{4}{5}\right) = 2.4 \text{ passangers}$$

$Ls = Ls_M + Ls_B = 2 + 2.4 = 4.4$ passangers

WL_8:

Data: λ = 5 clients/hour, arrivals follow a Poisson distribution; 1/ μ = 20 minutes $\Rightarrow \mu$ = 60/20 =3 clients/hour; Service time follows a negative exponential distribution

M=S= 3 $\Rightarrow$ M/M/3 model

a)

$$P_0 = \frac{1}{\left[\sum_{n=0}^{S-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{(\lambda/\mu)^S}{S!} \times \frac{S\mu}{S\mu - \lambda}}$$

$$P_0 = \frac{1}{\left(1 + \frac{5}{3} + \frac{1}{2} \times \left(\frac{5}{3}\right)^2\right) + \frac{1}{3!} \times \left(\frac{5}{3}\right)^3 \times \frac{3 \times 3}{3 \times 3 - 5}} = \frac{1}{(4.0556 + 0.7716 \times 2.25)} = 0.1727$$

$$Lq = \frac{\lambda \times \mu \times \left(\frac{\lambda}{\mu}\right)^S}{(S-1)!\,(S\mu - \lambda)^2} P_0$$

$L_q = \lambda \times W_q$

$$Wq = \frac{\mu \times \left(\frac{\lambda}{\mu}\right)^S}{(S-1)!\,(S\mu - \lambda)^2} P_0$$

$$W_q = \frac{3 \times \left(\frac{5}{3}\right)^3}{(3-1)!(3 \times 3 - 5)^2} \times 0{,}1727 = \frac{13.8889}{32} \times 0{,}1727 = 0.0750 \text{ hours} = 4.4974 \text{ minutes}$$

b)

$$\rho = \frac{\lambda}{S\mu} = \frac{5}{3 \times 3} = 0.5556$$

Time spent with customers = 5 × 8 × 0.5556 = 22.222 hours per week

c)

C_w= 1.5 euros/hour and C_H= 30 euros/day

Total cost = $\lambda \times 8 \times W_q \times C_w + 3 \times C_h$= 5 × 8 × 0.0750 × 1.5 + 3 × 30 = 94.5 euros/day

d)

λ = 14 clients/hour, arrivals follow a Poisson distribution

$S \times \mu > \lambda \Leftrightarrow S \times 3 > 14 \Leftrightarrow S > 4.67 \Rightarrow S = 5$

WL_9:

Data:

4 doctors (maximum)

Arrivals follow a Poisson distribution, according to the following pattern:

Hour	Arrival pattern per hour
From 9 to 15	6 patients
From 15 to 20	4 patients
From 20 to 24	12 patients

1/ μ = 12 minutes ⇒ μ = 5 patients/hour ⇒ service time follows a negative exponential distribution

Time in the Waiting Room is <u>less</u> than 5 minutes ⇒ Wq < 5 minutes

From 20 to 24: for M = 3, P_0 =5/89, and for M = 4, P_0 =25/301

From 9 to 15	λ = 6 patients/hour μ = 5 patients/hour

Sμ > λ ⇔ S > 6/5 ⇒ **S=2 ⇒ M/M/2**

$$P_0 = \frac{1}{\left(1+\frac{6}{5}\right)+\frac{1}{2!}\times\left(\frac{6}{5}\right)^2\times\frac{2\times 5}{2\times 5-6}} = 0.25$$

$$W_q = \frac{5\times\left(\frac{6}{5}\right)^2}{(2-1)!(2\times 5-6)^2}\times 0.25 = 0.1125 \text{ hours} = 6.75 \text{ minutes} > 5 \text{ minutes}$$

S=3 ⇒ M/M/3

P_0 = 0.2941

$$W_q = \frac{5\times\left(\frac{6}{5}\right)^3}{(3-1)!(3\times 5-6)^2}\times 0.2941 = 0.01568 \text{ hours} = 0.9411 \text{ minutes} < 5 \text{ minutes}$$

From 15 to 20	λ = 4 patients/hour μ = 5 patients/hour

$S\mu > \lambda \Leftrightarrow S > 4/5 \Rightarrow$ **S=1** $\Rightarrow$ **M/M/1**

$$W_q = \frac{\lambda}{\mu\,(\mu - \lambda)} = \frac{4}{5 \times 1} = 0.8\ hours = 48\ minutes > 5\ minutes$$

S=2 $\Rightarrow$ **M/M/2**

$$P_0 = \frac{1}{\left(1 + \frac{4}{5}\right) + \frac{1}{2!} \times \left(\frac{4}{5}\right)^2 \times \frac{2 \times 5}{2 \times 5 - 4}} = 0.4286$$

$$W_q = \frac{5 \times \left(\frac{4}{5}\right)^2}{(2-1)!(2 \times 5 - 4)^2} \times 0.4286 = 0.038\ \text{hours} = 2.28\ \text{minutes} < 5\ \text{minutes}$$

From 20 to 24	λ = 12 patients/hour μ = 5 patients/hour

$S\,\mu > \lambda \Leftrightarrow S > 12/5 \Rightarrow$ **S=3** $\Rightarrow$ **M/M/3**

P_0 = 5/89

$$W_q = \frac{5 \times \left(\frac{12}{5}\right)^3}{(3-1)!(3 \times 5 - 12)^2} \times (5/89) = 0.2157\ \text{hours} = 12.9438\ \text{minutes} > 5\ \text{minutes}$$

S=4 $\Rightarrow$ **M/M/4**

P_0 = 25/301

$$W_q = \frac{5 \times \left(\frac{12}{5}\right)^4}{(4-1)!(4 \times 5 - 12)^2} \times \left(\frac{25}{301}\right) = 0.0359\ \text{hours} = 2.1528\ \text{minutes} < 5\ \text{minutes}$$

Hour	λ	**Number of doctors on duty**
From 9 to 15	6 patients	**3**
From 15 to 20	4 patients	**2**
From 20 to 24	12 patients	**4**

WL_10:

Data:

λ = 30 trucks/hour, arrival follow a Poisson distribution; μ = 35 trucks/hour, service time follows a negative exponential distribution; Waiting cost for trucks (and drivers) = 18 Euros/truck/hour; Working hours = 16 hours per day (7 days a week); M/M/1 model

a) $L_s = \frac{\lambda}{(\mu-\lambda)} = \frac{30}{35-30} = 6\ trucks$

b) $W_s = L_s/\lambda$ **=** 6/30 = 0.2 hours = *12 minutes*

c) Utilization rate = ρ= λ/ μ = 30/35= 0.8571, or *85.71%*

d) $P(n \geq 3) = P(n > 2) = (30/35)^3$ = *0.6297*

e) W_s = 0.2 hours
Unloading Cost = 30 trucks/hour × 0.2 hours × 16 hours/day × 18 euros/hour= 1 728 euros/day

f) Total cost 2 weeks (without enlargement) = 1 728€/day × 14 days = €24 192,0 euros

Total cost for two weeks (with enlargement) = 9 000€ + (24 192,0/2) = €21 096

Based on the total costs, we can conclude that *it is worth enlarging the bin*.

WL_11:

a) M/M/3 model

λ = 12 cars/hour
$1/\mu$ = 12 minutes $\Rightarrow$ μ = 5 cars/hour
P_0 = 0.05618

$$Lq = \frac{\lambda \times \mu \times \left(\frac{\lambda}{\mu}\right)^S}{(S-1)!\,(S\mu - \lambda)^2} P_0$$

$$L_q = \frac{12 \times 5 \times \left(\frac{12}{5}\right)^3}{(3-1)!\,(3 \times 5 - 12)^2} \times 0.05618 = 2.589\ cars$$

b)

S= 3, n = 4 $\Rightarrow$ n > S $\Rightarrow$ $P_n = \frac{\left(\frac{\lambda}{\mu}\right)^n}{S!S^{n-S}} P_0$

$$P_4 = \frac{\left(\frac{12}{\mu}\right)^4}{3!\,3^{4-3}} \times 0.05618 = 0.1036$$

c)

$$\rho = \frac{\lambda}{S\mu} = \frac{12}{3 \times 5} = 0.8$$

WL_12:

a)
Arrivals at CARIMBADO

λ = 20 clients/hour $\Rightarrow$ Arrivals at the Personalized service λ = 5 clients/hour

M/M/1 model, with μ= (1/0.1) = 10 clients/hour

1-P_0 = ρ = (5/10) = 0.5, or *50%*

b)
Self-service counter $\Rightarrow$ λ = 15 clients/hour

M/D/1 model

λ = 15 clients/hour

Wq = 5 minutes $\Rightarrow$ L_q = λ Wq = 15*(5/60) = *1.25 clients*

c)
Ws = 1/(μ-λ) = 1/(10-5) hours = *12 minutes*

WL_13:

Arrival rate = λ = 0.706 ajustments/ computer/hr.
Service rate μ = 1 computer / 15 min. = 4 computers/hr.
N = 5; one server
M/M/1/5

a)

$$P_0 = \frac{1}{\sum_{n=0}^{N} \frac{N!}{(N-n)!}\left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{\sum_{n=0}^{5} \frac{5!}{(5-n)!}\left(\frac{.706}{4}\right)^n}$$

$$= \frac{1}{1(.1765)^0 + 5(.1765)^1 + 20(.1765)^2 + 60(.1765)^3 + 120(.1765)^4 + 120(.1765)^5}$$

$$= \frac{1}{1 + 0.8825 + 0.6230 + 0.3299 + 0.1165 + 0.0206} = \frac{1}{2.9725} = 0.3364$$

$$L_q = N - \left(\frac{\lambda+\mu}{\lambda}\right)(1-P_0) = 5 - \left(\frac{.706+4}{.706}\right)(1-.3364) = 5 - 4.423 = 0.577 \text{ computers}$$

(b) $L_s = L_q + \bar{\lambda}/\mu = 0.577 + 0.6636 = 1.241$ computers

$P_0 = 1 - (\bar{\lambda}/\mu) \Rightarrow 1 - P_0 = \bar{\lambda}/\mu = 0.6636$

MULTIPLE CHOICE QUESTIONS

1. CANNEDHEAT produces canned tuna using 25 canning machines. On average, each machine breaks-down 2 times per hour, following a Poisson process. CANNEDHEAT employs 5 maintenance technicians. The repair time follows a negative exponential distribution, with an average of 6 minutes. The utilization rate of the technicians is 80%, and P0 = 0.0085.

What is the probability of all 4 machines being broken down at the same time?		
1	X	0.172
2		0.20
3		0.80
4		0.828

What is the average time a machine has to wait to be repaired?		
1		7.5 minutes
2		6 minutes
3	X	1.5 minutes
4		1.03 minutes

2. OKCOMPUTER, a data processing firm, has 4 backup servers. On average, each server reports one breakdown per day, according to a Poisson process. The firm employs 2 maintenance technicians. Each technician needs 4 hours to complete a repair. Repair time follows a negative exponential distribution. The probability of having the technicians free is of 18.39%. The firm operates 8 hours per day.

What is the probability of all 4 backup servers being in repair status?		
1		31.72%
2		27.69%
3	x	1.15%
4		3.45%

If P1=36.78%, P2=27.59%, and P3=13.79%, how many backup servers are repaired per day?		
1	X	2.5 backup servers
2		0.3 backup servers
3		1.5 backup servers
4		2.0 backup servers

3. Clients arrive at the ticket office of MetroLx at an average rate of 27 per hour (according to a Poisson distribution). The ticket office employs one worker whom, on average, takes 2 minutes to serve a client. Servicing time follows a negative exponential distribution.

What is the average number of clients waiting in line to purchase a ticket?		
1		30 clients
2		9 clients
3	X	8.1 clients
4		7 clients

What is the probability of having, at any given moment in time, more than 3 clients on the ticket office?		
1	X	65.61%
2		72.90%
3		53.14%
4		59.05%

What is the likelihood of a client having to wait to be served?		
1	X	90.0%
2		10.0%
3		50.0%
4		81.0%

(Adapted from quiz 2 2012/2013)

4. Six students arrive, on average, every hour to the reception desk of S. Bernard's high school administrative office in accordance with a *Poisson* process. Only one clerk works at administrative office, which led to complaints that the average time students spend in the office, which is 5 minutes, is excessive. Please assume that service time follows a negative exponential distribution.

How long does it take, on average, a student being served by the office clerk?		
1		5 minutes
2		18 minutes
3	x	3.33 minutes
4		1,67 minutes

Assuming the average number of students served in one hour is μ=8 students, what is the probability of finding more than 2 students in line at any given moment?		
1		10.55%
2	x	31.64%
3		14.06%
4		42.19%

(Adapted from quiz 2 2013/2014)

5. A new customer arrives Mrs Bina Grocery store every 3 minutes. Clients wait for their turn in a single line. Two employees work at the grocery store helping each other: one as a cashier (register the purchases) and other in the packaging of the groceries. This process allows for a service of 25 customers per hour. Assume arrivals follow a *Poisson* distribution and that service follows a negative exponential distribution.

What is the average number of customers waiting in queue to pay for their groceries?		
1		4 clients
2		0.15 clients
3	x	3.2 clients
4		0.016 clients

(Adapted from quiz 2 2013/2014)

6. The pharmacy GOODHEALTH employs two workers. As clients arrive, they wait on a single queue for the first available clerk to serve them. Average hourly arrivals are 18 and follow a Poisson distribution. Service time is, on average, of 6 minutes and follows a negative exponential distribution. It is known that the probability of the system being empty is equal to 5.26%.

What is the utilisation rate of the system?		
1		77.15%
2		94.74%
3		83.8%
4	x	90%

What is the average time a client needs to wait before being served?		
1		1.85 hours
2		0.52 hours
3	x	25.56 minutes
4		7.67 minutes