Financial Economics

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Homework 2: Solutions

Problem 1.

The representative consumer maximizes a CRRA utility function.

$$E_t \sum \beta^j c_{t+j}^{1-\gamma}.$$

Consumption is given by an endowment stream.

(a) Show that with log utility, the price/consumption ratio of the consumption stream is constant, no matter what the distribution of consumption growth.

Solution:

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t} \right)^{-\gamma} c_{t+j}$$

$$\frac{p_t}{c_t} = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t} \right)^{1-\gamma}$$

If $\gamma = 1$,

$$\frac{p}{c} = \left(\beta + \beta^2 + ... \right) = \frac{\beta}{1 - \beta}$$

(b) Suppose there is news at time t that future consumption will be higher. For $\gamma < 1$, $\gamma = 1$, and $\gamma > 1$, evaluate the effect of this news on the price. Make sense of your results. (Note: there is a real-world interpretation here. It's often regarded as a puzzle that the market declines on good economic news. This is attributed to an expectation by the market that the Fed will respond to such news by raising interest rates. Note that $\gamma > 0$ in this problem gives a completely real and frictionless interpretation to this phenomenon!)

Solution: If $\gamma < 1$ then a rise in c_{t+j} raises p_t. If $\gamma > 1$, however, a rise in c_{t+j} lowers p_t. Any piece of news has two possible effects: cashflows and discount rates. Higher future consumption means you'd like to eat more now, so rates of return must rise to keep you from doing so. If $\gamma > 1$, the discount rate rises faster than the payoffs, so the price actually declines.

$\gamma > 1$ is the typical parameter, so this is the case we should expect. Many commentators think that it's puzzling that news of good economic growth often sends stock prices down. They blame the Fed, i.e. that good growth news will make the Fed tighten which is bad for the economy. This problem shows you that good growth news / higher interest rates / lower stock prices is exactly what you expect from a completely frictionless model.

Problem 2.

The linear quadratic permanent income model is a very useful general equilibrium model that we can solve in closed form. It specifies a production technology rather than fixed endowments, and it easily allows aggregation of disparate consumers.

The consumer maximizes

$$E \sum_{t=0}^{\infty} \beta^t \left(-\frac{1}{2}\right) (c_t - c^*)^2$$

subject to a linear technology

$$k_{t+1} = (1+r)k_t + i_t$$

$$i_t = e_t - c_t$$

e_t is an exogenous endowment or labor income stream. Assume $\beta = 1/(1+r)$; the discount rate equals the interest rate or marginal productivity of capital.

(a) Show that optimal consumption follows

$$c_t = rk_t + r\beta \sum_{j=0}^{\infty} \beta^j E_t e_{t+j}$$

$$c_t = c_{t-1} + (E_t - E_{t-1}) r\beta \sum_{j=0}^{\infty} \beta^j E_t e_{t+j}$$

i.e., consumption equals permanent income, precisely defined, and consumption follows a random walk whose innovations are equal to innovations in permanent income.

Solution:

The Lagrangian is

$$L = E \left\{ \sum_{t=0}^{\infty} \beta^t \left(-\frac{1}{2} \right) (c_t - c^*)^2 + \sum_{t=0}^{\infty} \lambda_t \left(k_{t+1} - (1+r)k_t + c_t - e_t \right) \right\}$$

The first order conditions are

$$\beta^t (c_t - c^*) = \lambda_t$$

$$(1+r) E_t \lambda_{t+1} = \lambda_t$$

From these two conditions we get

$$c_t - c^* = (1+r)\beta E_t (c_{t+1} - c^*)$$

with $\beta = 1/(1+r)$

$$c_t = E_t c_{t+1}$$

Consumption follows a random walk.

Define $R = (1+r)$. Iterate the technology forward,

$$k_{t+2} = R(Rk_t + i_t) + i_{t+1} = R^2 k_t + Ri_t + i_{t+1}$$

$$k_{t+3} = R^3 k_t + R^2 i_t + Ri_{t+1} + i_{t+2}$$

$$\frac{1}{R^3} k_{t+3} = k_t + \frac{1}{R}\left[i_t + \frac{1}{R} i_{t+1} + \frac{1}{R^2} i_{t+2}\right]$$

$$\beta^3 k_{t+3} = k_t + \beta \left[i_t + \beta i_{t+1} + \beta^2 i_{t+2} \right]$$

Continuing and with the transversality condition $lim_{T \to \infty} \beta^T k_{t+T} = 0$, and $i = e - c$

$$k_t + \sum_{j=0}^{\infty} \beta^{j+1} e_{t+j} = \sum_{j=0}^{\infty} \beta^{j+1} c_{t+j}$$

Taking expectations,

$$k_t + \sum_{j=0}^{\infty} \beta^{j+1} E_t e_{t+j} = \sum_{j=0}^{\infty} \beta^{j+1} E_t c_{t+j}$$

Intuitively, the present value of future consumption must equal wealth plus the present value of future endowment (labor income).

Now, substitute the first order condition in the budget constraint.

$$\begin{aligned} k_t + \sum_{j=0}^{\infty} \beta^{j+1} E_t e_{t+j} &= \sum_{j=0}^{\infty} \beta^{j+1} c_t \\ \frac{\beta}{1-\beta} c_t &= \frac{1}{R} \frac{1}{1-\frac{1}{R}} c_t \\ \frac{1}{R-1} c_t &= \frac{c_t}{r} \end{aligned}$$

$$c_t = r k_t + r \sum_{j=0}^{\infty} \beta^{j+1} E_t e_{t+j}$$

Consumption equals the annuity value of wealth (capital) $r k_t$ plus the present value of future labor income (endowment). This is the *permanent income hypothesis*.

Now to the random walk in consumption. Just quasi-first difference, and use $k_{t+1} -$

$$k_t = rk_t + i_t.$$

$$
\begin{aligned}
c_t &= rk_t + r\left(\beta e_t + \beta^2 E_t e_{t+1} + \beta^3 E_t e_{t+2} + ...\right) \\
c_{t-1} &= rk_{t-1} + r\left(\beta e_{t-1} + \beta^2 E_{t-1} e_t + \beta^3 E_{t-1} e_{t+1} + ...\right) \\
c_t - c_{t-1} &= r\left(k_t - k_{t-1}\right) + ... \\
c_t - c_{t-1} &= r\left(rk_{t-1} + e_{t-1} - c_{t-1}\right) + ... \\
c_t - c_{t-1} &= r\left(rk_{t-1} + e_{t-1} - rk_{t-1} - r\sum_{j=0}^{\infty} \beta^{j+1} E_{t-1} e_{t-1+j}\right) + ... \\
c_t - c_{t-1} &= re_{t-1} + r\left(\beta e_t + \beta^2 E_t e_{t+1} + \beta^3 E_t e_{t+2} + ...\right) \\
&\quad - \left(r^2 + r\right)\left(\beta e_{t-1} + \beta^2 E_{t-1} e_t + \beta^3 E_{t-1} e_{t+1} + ...\right) \\
c_t - c_{t-1} &= re_{t-1} + r\left(\beta e_t + \beta^2 E_t e_{t+1} + \beta^3 E_t e_{t+2} + ...\right) \\
&\quad -r\left(e_{t-1} + \beta E_{t-1} e_t + \beta^2 E_{t-1} e_{t+1} + ...\right) \\
c_t &= c_{t-1} + \left(E_t - E_{t-1}\right) r\beta \sum_{j=0}^{\infty} \beta^j e_{t+j}
\end{aligned}
$$

Consumption is a random walk. We knew that from the first order conditions, $c_t = E_t c_{t+1}$. With a full equilibrium model we can now relate the innovations to consumption to fundamental shocks to technology. In this model, changes in consumption equal the innovation in the present value of future income.

(b) Assume that the endowment e_t follows an AR(1) $e_t = \rho e_{t-1} + \varepsilon_t$. Calculate and interpret the result for $\rho = 1$ and $\rho = 0$. (The result looks like a "consumption function" relating consumption to capital and current income, except that the slope of that function depends on the persistence of income shocks. Transitory shocks will have little effect on consumption, and permanent shocks a larger effect.).

Solution:

$$
\begin{aligned}
c_t &= rk_t + r\sum_{j=0}^{\infty}\beta^{j+1}E_t e_{t+j} \\
&= rk_t + r\beta\sum_{j=0}^{\infty}\beta^j\rho^j e_t \\
&= rk_t + \frac{r\beta}{1-\beta\rho}e_t
\end{aligned}
$$

This equation does look like a consumption function, but notice that the parameter relating consumption c to income e depends on the persistence of income e. It is not a "psychological law" or a constant of nature. If the government changes policy so that income is more unpredictable (i.e. it gets rid of the predictable part of recessions), then this coefficient declines dramatically.

The income coefficient is not "policy-invariant." This is the basis of Bob Lucas' (1974) dramatic deconstruction of Keynesian models based on consumption functions that were used for policy experiments.

$$
\begin{aligned}
c_t &= c_{t-1} + (E_t - E_{t-1})\, r\beta \sum_{j=0}^{\infty} \beta^j e_{t+j} \\
&= c_{t-1} + r\beta \sum_{j=0}^{\infty} \beta^j \rho^j e_t - r\beta \sum_{j=0}^{\infty} \beta^j \rho^{j+1} e_{t-1} \\
&= c_{t-1} + r\beta \sum_{j=0}^{\infty} \beta^j \rho^j \left(e_t - \rho e_{t-1}\right) \\
&= c_{t-1} + r\beta \sum_{j=0}^{\infty} \beta^j \rho^j \varepsilon_t \\
&= c_{t-1} + \frac{r\beta}{1-\rho\beta} \varepsilon_t
\end{aligned}
$$

In both equations, you see that consumption responds to “permanent income” and that as shocks get more “permanent” — as ρ rises — consumption moves more.

(c) Calculate the one period interest rate (it should come out to r of course) and the price of a claim to the consumption stream. e and k are the only state variables, so the price should be a function of e and k. Interpret the time-variation in the price of the consumption stream. (This consumer gets more risk averse as consumption rises to c^*. c^* is the bliss point, so at the bliss point there is no average return that can compensate the consumer for greater risk.)

Solution:

R was the rate of return on technology. Despite the symbol, it is not (yet) the interest rate — the equilibrium rate of return on one-period claims to consumption. That remains to be proved. The logic is, first find c, then price things from the equilibrium consumption stream. To be precise and pedantic, call the risk free rate R_f, and

$$\begin{aligned}
\frac{1}{R_f} &= E_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)}\right) \\
&= E_t\left(\beta \frac{(c^* - c_{t+1})}{(c^* - c_t)}\right) \\
&= \beta \frac{(c^* - c_t)}{(c^* - c_t)} = \beta = \frac{1}{R}
\end{aligned}$$

Now, the fun stuff. We can approach the price of the consumption stream by brute force,

$$
\begin{aligned}
p_t &= E_t \sum_{j=1}^{\infty} m_{t,t+j} c_{t+j} = E_t \sum_{j=1}^{\infty} \beta^j \frac{\left(c^* - c_{t+j}\right)}{(c^* - c_t)} c_{t+j} \\
&= E_t \sum_{j=1}^{\infty} \beta^j \frac{\left(c^* c_{t+j} - c_{t+j}^2\right)}{(c^* - c_t)} = \sum_{j=1}^{\infty} \beta^j \frac{\left(c^* c_t - E_t\left(c_{t+j}^2\right)\right)}{(c^* - c_t)} \\
&= \sum_{j=1}^{\infty} \beta^j \frac{\left(c^* c_t - c_t^2 - var_t\left(c_{t+j}\right)\right)}{(c^* - c_t)}
\end{aligned}
$$

$$
c_{t+1} = c_t + \frac{r\beta}{1 - \rho\beta} \varepsilon_{t+1}
$$

$$c_{t+2} = c_{t+1} + \frac{r\beta}{1-\rho\beta}\varepsilon_{t+2} = c_t + \frac{r\beta}{1-\rho\beta}\left(\varepsilon_{t+1} + \varepsilon_{t+2}\right)$$

$$c_{t+j} = c_t + \frac{r\beta}{1-\rho\beta}\left(\varepsilon_{t+1} + \varepsilon_{t+2} + ... + \varepsilon_{t+j}\right)$$

$$E_t\left(c_{t+j}\right) = c_t$$

$$var_t\left(c_{t+j}\right) = j\left(\frac{r\beta}{1-\rho\beta}\right)^2 \sigma_\varepsilon^2$$

$$
\begin{aligned}
p_t &= \sum_{j=1}^{\infty} \beta^j \frac{c_t\left(c^* - c_t\right) - j\left(\frac{r\beta}{1-\rho\beta}\right)^2 \sigma_\varepsilon^2}{\left(c^* - c_t\right)} \\
&= \sum_{j=1}^{\infty} \beta^j \left[c_t - \frac{j\left(\frac{r\beta}{1-\rho\beta}\right)^2 \sigma_\varepsilon^2}{\left(c^* - c_t\right)}\right] \\
&= \sum_{j=1}^{\infty} \beta^j c_t - \sum_{j=1}^{\infty} j\beta^j \frac{\left(\frac{r\beta}{1-\rho\beta}\right)^2 \sigma_\varepsilon^2}{\left(c^* - c_t\right)}
\end{aligned}
$$

using $\sum_{j=1}^{\infty} j\beta^j = \frac{\beta}{(\beta-1)^2}$ get

$$
\begin{aligned}
p_t &= \frac{\beta}{1-\beta}c_t - \frac{\beta}{(\beta-1)^2}\frac{\left(\frac{r\beta}{1-\rho\beta}\right)^2\sigma_\varepsilon^2}{(c^* - c_t)} \\
&= \frac{\frac{1}{1+r}}{1-\frac{1}{1+r}}c_t - \frac{\frac{1}{1+r}}{\left(\frac{1}{1+r}-1\right)^2}\frac{\left(\frac{r\beta}{1-\rho\beta}\right)^2\sigma_\varepsilon^2}{(c^* - c_t)} \\
&= \frac{c_t}{r} - \frac{\beta}{(1-\rho\beta)^2}\frac{\sigma_\varepsilon^2}{(c^* - c_t)}
\end{aligned}
$$

The first term is the risk-neutral price — the value of a perpetuity paying c. (Don't forget $E_t\left(c_{t+j}\right) = c_t$) The second term is a risk correction. It lowers the price. If σ_ε is high — more risk —the price is lower. If ρ is high — more persistent consumption — the price is lower.

Now, the hard term — the effect of consumption. At the bliss point, the consumer is as happy as can be, and marginal utility falls to zero. Hence, the consumer is infinitely risk averse. ($\frac{u''(c)}{u'(c)}$ rises to infinity). There is no consumption you can give him to compensate for risk, since he's at the bliss point. No surprise that the price goes off to $-\infty$ here. As consumption rises towards the bliss point, the consumer gets more and more risk averse (u'' is constant, u' is falling), so the price declines. Above the bliss point, the consumer values consumption negatively, so the price is higher than the risk-neutral version. This feature — that risk aversion rises as consumption rises — is obviously not a good one. Quadratic utility is best used as a local approximation. If you use a quadratic model, find a c^* that gives a sensible risk aversion, and then make sure the model doesn't get too far away! (The CAPM is a quadratic model.)

Problem 3. This is not only a historically important model, it introduces a very important method. Evaluating infinite sums as in the last problem is a huge pain. In most models, conditioning information is a function of only a few state variables, x_t. Everything you could want to know about the current state of the economy, and the conditional distribution of everything you could want to know in the future is contained in the state variables. Hence, prices (at least properly scaled) have to be a function of the state variables. Instead of solving for p in terms of a huge infinite sum, you can solve the functional equation $p(x_t) = E_t[m_{t+1}(x_t, x_{t+1})(p(x_{t+1})+d_{t+1})]$, (an equation that specifies a function in implicit form).

Consider again CRRA utility,

$$E_t \sum \beta^j c_{t+j}^{1-\gamma}$$

Consumption growth follows a two-state Markov process. The states are $\Delta c_t = c_t/c_{t-1} = h, l$, and a 2×2 matrix Π governs the set of transition probabilities, i.e. $pr(\Delta c_{t+1} = h | \Delta c_t = l) = \pi_{l \to h}$. (This is the Mehra-Prescott 1986 model)

(a) Find the riskfree rate (price of a certain real payoff of one) in this economy. This price is generated by $p_t^b = E_t(m_{t,t+1}1)$. You are looking for two values, the price in the l state and the price in the h state.

Solution: From the basic first order condition,

$$p_t^b = E_t\left(\beta \frac{u'(c_{t+1})}{u'(c_t)} 1\right) = E_t\left[\beta \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma}\right]$$

$$p^b(\Delta c_t = h) = \beta \pi_{h \to h} h^{-\gamma} + \beta \pi_{h \to l} l^{-\gamma}$$

$$p^b(\Delta c_t = l) = \beta \pi_{l \to h} h^{-\gamma} + \beta \pi_{l \to l} l^{-\gamma}$$

$$\begin{bmatrix} p^b(\Delta c_t = h) \\ p^b(\Delta c_t = l) \end{bmatrix} = \begin{bmatrix} \pi_{h \to h} & \pi_{h \to l} \\ \pi_{l \to h} & \pi_{l \to l} \end{bmatrix} \begin{bmatrix} \beta h^{-\gamma} \\ \beta l^{-\gamma} \end{bmatrix}$$

$$\mathbf{p}^b = \beta \Pi \left(\mathbf{x}^{-\gamma} \right)$$

as we assumed it is a function of the current state

$$p^b(x_t) = \beta \Pi(x_t)$$

The riskfree rate is of course

$$R_f = \frac{1}{p^b}$$

$$R_f\left(h\right) = \frac{1}{\beta\pi_{h\rightarrow h}h^{-\gamma} + \beta\pi_{h\rightarrow l}l^{-\gamma}}$$

$$R_f\left(l\right) = \frac{1}{\beta\pi_{l\rightarrow h}h^{-\gamma} + \beta\pi_{l\rightarrow l}l^{-\gamma}}$$

another problem: Find the value of a perpetuity, i.e. a bond that pays a coupon of one per period forever.

Solution:

The value of a perpetuity at any date is

$$p_t^p = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} \times 1 = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} \times 1$$

A direct attack leads to a mountain of algebra. Instead, think about the one period relation

$$p_t^p = E_t \left(\beta \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} \left(1 + p_{t+1}^p\right) \right)$$

All information about the future in this economy is contained in the current value of consumption growth. (Consumption growth is a state variable for this economy.) Thus p_t^p and p_{t+1}^p can only depend on consumption growth at t and $t+1$ respectively. p_t^p can only take on two values, $p_t^p(h)$ in the good state and $p_t^p(l)$ in the bad state. We can find these prices as follows

$$p^p(h) = \beta \pi_{h\to h} h^{-\gamma} [1 + p^p(h)] + \beta \pi_{h\to l} l^{-\gamma} [1 + p^p(l)]$$

$$p^b(l) = \beta \pi_{l\to h} h^{-\gamma} [1 + p^p(h)] + \beta \pi_{l\to l} l^{-\gamma} [1 + p^p(l)]$$

This is a system of equations in the two unknowns $p^p(h)$ and $p^b(l)$. Thus, it is easy to solve for $p^p(h)$ and $p^b(l)$.

(b) To model a stock, we can think of an asset that pays consumption as its dividend. The value of the whole Portuguese economy (think of it as a big corporation) is the value of a claim to the consumption it can provide us. Now, the stock price

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{u'(c_{t+j})}{u'(c_t)} c_{t+j} = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{c_{t+j}}{c_t}\right)^{-\gamma} c_{t+j}$$

depends on the level of consumption, not just the growth rate at each date. However, we can apply the same trick as for the perpetuity to the ratio of price to consumption, i.e. the price/dividend ratio of the stock,

$$p_t = E_t[m_{t+1}(p_{t+1} + c_{t+1})]$$

$$\frac{p_t}{c_t} = E_t[m_{t+1}(\frac{p_{t+1}}{c_{t+1}} + 1)\frac{c_{t+1}}{c_t}]$$

$$\frac{p_t}{c_t} = E_t[\beta(\frac{p_{t+1}}{c_{t+1}} + 1)\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma}]$$

$\frac{p}{c}$ can take on only two values, $\frac{p}{c}(h)$ and $\frac{p}{c}(l)$. Proceed as with the perpetuity to find those two values.

Solution:

The stock is exactly like the perpetuity but with $h^{1-\gamma}$ where there was $h^{-\gamma}$ and so forth

$$\frac{p_t}{c_t} = E_t[\beta(\frac{p_{t+1}}{c_{t+1}} + 1)\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma}]$$

$$\frac{p}{c}(h) = \beta\pi_{h\to h}h^{1-\gamma}\left[1+\frac{p}{c}(h)\right] + \beta\pi_{h\to l}l^{1-\gamma}\left[1+\frac{p}{c}(l)\right]$$

$$\frac{p}{c}(l) = \beta\pi_{l\to h}h^{1-\gamma}\left[1+\frac{p}{c}(h)\right] + \beta\pi_{l\to l}l^{1-\gamma}\left[1+\frac{p}{c}(l)\right]$$

$$\begin{bmatrix} 1-\beta\pi_{h\to h}h^{1-\gamma} & -\beta\pi_{h\to l}l^{1-\gamma} \\ -\beta\pi_{l\to h}h^{1-\gamma} & 1-\beta\pi_{l\to l}l^{1-\gamma} \end{bmatrix} \begin{bmatrix} \frac{p}{c}(h) \\ \frac{p}{c}(l) \end{bmatrix} = \begin{bmatrix} \beta\pi_{h\to h}h^{1-\gamma} + \beta\pi_{h\to l}l^{1-\gamma} \\ \beta\pi_{l\to h}h^{1-\gamma} + \beta\pi_{l\to l}l^{1-\gamma} \end{bmatrix}$$

$$\begin{bmatrix} \frac{p}{c}h \\ \frac{p}{c}l \end{bmatrix} = \begin{bmatrix} 1-\beta\pi_{h\to h}h^{1-\gamma} & -\beta\pi_{h\to l}l^{1-\gamma} \\ -\beta\pi_{l\to h}h^{1-\gamma} & 1-\beta\pi_{l\to l}l^{1-\gamma} \end{bmatrix}^{-1} \begin{bmatrix} \beta\pi_{h\to h}h^{1-\gamma} + \beta\pi_{h\to l}l^{1-\gamma} \\ \beta\pi_{l\to h}h^{1-\gamma} + \beta\pi_{l\to l}l^{1-\gamma} \end{bmatrix}$$

(c) Pick $\beta = 0.99$ and try $\gamma = 0.5, 5$ (Try more if you feel like it). Calibrate the consumption process to have a 1% mean and 1% standard deviation, and consumption growth uncorrelated over time. Calculate prices and returns in each state.

Solution:

Start with the calibration. It's most natural to take the two points to be equally above and below the mean, $h = 1.01 + z$, $l = 1.01 - z$ and equal probabilities.

(You could try different alternatives for z and for the probabilities.)

The expected value of the consumption growth has to be 1%

$$1/2\,(1.01 + z) + 1/2\,(1.01 - z) = 1.01$$

the standard deviation has to be 1%

$$1/2\,(z)^2 + 1/2\,(z)^2 = 1.01^2$$

implies $z = 0.01$.

Have to find stock returns.

$$R_{t+1} = \frac{p_{t+1} + c_{t+1}}{p_t} = \frac{(\frac{p_{t+1}}{c_{t+1}} + 1)\frac{c_{t+1}}{c_t}}{\frac{p_t}{c_t}}.$$

For each one have to get four values, the return from each state h and l to states h and l.

$$R_{t+1}\,(h \rightarrow l) = \frac{(\frac{p_{t+1}}{c_{t+1}}\,(l) + 1)\frac{c_{t+1}}{c_t}\,(l)}{\frac{p_t}{c_t}\,(h)}$$

and so on. After we have to compute the different equilibrium values,

		in	state
	to state	h	l
$\gamma = 0.5$			
p^b		0.985	0.985
R^f		1.5%	1.5%
p/c		196	196
R	h	2.52%	2.52%
	l	0.51%	0.51%

The first thing you notice is that all the prices and other forward-looking things are the same in each state. Thus the bond price, stock p/c ratio, risk free rate and expected returns are constant through time. Well, of course. Since the probabilities of h vs. l at $t+1$ are the same, everything looks the same going forward at t, whether you're in h or l at time t. Returns vary of course. If you go from l to h, you get a higher dividend and, since p/c is constant, a higher price too. Thus return is good to h and bad to l. Since the bond price never changes the bond return is constant, and equal to the risk free rate.

(Verify that the stock expected return is a little more than the risk free rate, but not much. Use expressions like

$$E_t\left(R_{t+1}|h\right) = \pi_{h\to h}\frac{(\frac{p_{t+1}}{c_{t+1}}(h)+1)h}{\frac{p_t}{c_t}(h)} + \pi_{h\to l}\frac{(\frac{p_{t+1}}{c_{t+1}}(l)+1)l}{\frac{p_t}{c_t}(h)}$$

to compute the stock expected return)

It is not surprising that the equity premium is small in this model since this model is in fact the original model that launched the "equity premium puzzle," and its inability to generate a large — 6% or more — risk premium for stocks is the central puzzle.

For $\gamma = 5$ the table is:

	to state	in h	state l
$\gamma = 5$			
p^b		0.943	0.943
R^f		6.01%	6.01%
p/c		19.96	19.96
R	h	7.11%	7.11%
	l	5.01%	5.01%

The risk free rate is higher since $R_f \approx 1 + \delta + \gamma E_t \Delta c_{t+1}$. As expected an higher γ gives an higher equity premium, since $\log(\text{equity premium}) = \gamma\sigma^2$. Still cannot get the total equity premium in the data (6%). (*Verify this*) However, the mean stock return is still almost exactly the same as the riskfree rate. Also, stock returns are perfectly correlated with consumption growth. The standard deviation of stock returns is about 1%, not about 20%. The Sharpe ratio $\left[E\left(R\right)-R_f\right]/\sigma\left(R\right)$ is way too low. (*Verify*)

(d) Now introduce serial correlation in consumption growth with $\gamma = 5$. (You can do this by adding weight to the diagonal entries of the transition matrix Π.) What effect does this have on the model?

Solution: To get serial correlation in consumption growth, we could try Π of the form

$$\Pi = \begin{bmatrix} 1/2 + \theta & 1/2 - \theta \\ 1/2 - \theta & 1/2 + \theta \end{bmatrix}$$

Now,

$$\begin{aligned} E_t \left(\Delta c_{t+1} | \Delta c_t = h \right) &= (1/2 + \theta)(g + z) + (1/2 - \theta)(g - z) \\ &= g + 2\theta z \end{aligned}$$

$$E_t\left(\Delta c_{t+1}|\Delta c_t = l\right) = (1/2-\theta)(g+z)+(1/2+\theta)(g-z)$$
$$= g - 2\theta z$$

Here are the results for a positive serial correlation.

		in	state
	to state	h	l
$\gamma = 5, \theta = 0.1$			
p^b		0.934	0.953
R^f		7.07%	4.97%
p/c		19.93	20.3
R	h	7.12%	5.05%
	l	6.99%	4.92%

In this case there is variation in prices with the initial state. In the previous case, the world looks the same from any starting date, so there is no variation in prices (ex-ante). The interest rate and stock return are higher from the high state, because expected future consumption growth is higher. Higher return means lower price or p/c.

The interest rate is higher in the good state than the bad state. In the good state it is more likely that tomorrow will be good as well, so $E_t \Delta c_{t+1}$ is higher, and so $R_f \approx 1 + \delta + \gamma E_t \Delta c_{t+1}$ is higher. Since the interest rate is higher in the good state, the bond price is lower. It's only a bit lower, since the interest rate is expected to revert to its mean pretty quickly.

The literature has offered many alternative models that do better in capturing the equity premium. The "habit persistence" utility described in Chapter 21 is one of several fundamental changes to this model that does the trick.

Homework

In the context of Problem 3 for the parameterization (d) verify if the expected excess perpetuity return is negative. That is if $E\left(R^p\right) - R^f < 0$.