Lisbon School of Economics & Management
Universidade de Lisboa

STATISTICS I

Economics / Finance/ Management

2nd Year/2nd Semester

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LESSON 9

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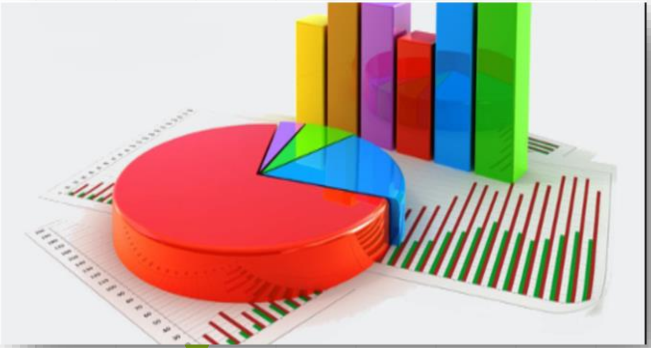

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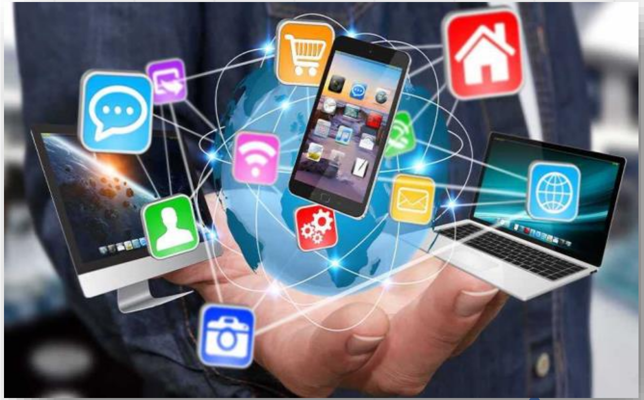

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Roadmap:

- Probability
- Random variable and two dimensional random variables:
    - Distribution
    - Joint distribution
    - Marginal distribution
    - Conditional distribution functions
- Expectations and parameters for a random variable and two dimensional random variables
- Discrete Distributions
- Continuous Distributions

Bibliography: Miller & Miller, John E. (2014) Freund's Mathematical Statistics with applications, 8th Edition, Pearson Education, [MM]

Two-Dimensional Continuous Variables: Exercises

Expectations and Parameters for two Dimensional Random Variables

Chapter 5

1

2. Let X and Y be two random variables such that

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & x>0, y>0 \\ 0, & otherwise \end{cases}$$

Compute $E(XY)$ and $E(X)$.

Exercise 2

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & x > 0, y > 0 \\ 0, & otherwise \end{cases}$$

$$E[g(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$g(x,y) = XY$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y) = \int_{0}^{+\infty}\int_{0}^{+\infty} xy\, e^{-x-y}\, dx\, dy =$$

$$= \int_0^{+\infty} ye^{-y}\int_0^{+\infty} xe^{-x}\, dx\, dy = \int_0^{+\infty} ye^{-y} \lim_{a\to+\infty} [-e^{-x}(x+1)]_0^a\, dy =$$

$$= \int_0^{+\infty} ye^{-y} \lim_{a\to+\infty} (-e^{-a}(a+1) - (\underbrace{-e^{0}(0+1)}_{1}))\, dy =$$

$$= \int_0^{+\infty} ye^{-y} \Big(\lim_{a\to+\infty}(\underbrace{-ae^{-a}}_{\frac{a}{e^a} \to 0}) - \underbrace{\lim_{a\to+\infty} e^{-a}}_{\frac{1}{e^a} \to 0} + 1 \Big)\, dy$$

$$= \int_0^{+\infty} ye^{-y}\, dy = \lim_{b\to+\infty} [-e^{-y}(y+1)]_0^b =$$

$$= \lim_{b\to+\infty} (-e^{-b}(b+1) - (\underbrace{-e^{0}(0+1)}_{1})) = \underbrace{\lim_{b\to+\infty} (-be^{-b})}_{0} - \underbrace{\lim_{b\to+\infty} e^{-b}}_{0} + 1 =$$

$$= 1$$

Exercise 2

$$\boxed{\int (u v') = u v - \int (v u')}$$

$u = x \qquad u' = 1$

$v' = e^{-x} \qquad v = \int v' = -e^{-x}$

$$\int x e^{-x} = x(-e^{-x}) - \int -e^{-x}(1) =$$

$$= -xe^{-x} + \int e^{-x} = -xe^{-x} - e^{-x} =$$

$$= -e^{-x}(1+x)$$

Exercise 2

$g(X,Y) = X$

$$E(X) = \int_0^{+\infty}\int_0^{+\infty} x \, f_{X,Y}(x,y)\,dy\,dx = \int_0^{+\infty}\int_0^{+\infty} x e^{-x-y}\,dy\,dx =$$

$$= \int_0^{+\infty} x e^{-x} \int_0^{+\infty} e^{-y}\,dy\,dx = \int_0^{+\infty} x e^{-x} \lim_{a \to +\infty} \left[-e^{-y}\right]_0^a dx =$$

$$= \int_0^{+\infty} x e^{-x} \lim_{a \to +\infty} \left(-e^{-a} - (-e^{0})\right) dx = \int_0^{+\infty} x e^{-x} \Big(\underbrace{\lim_{a \to +\infty} (-e^{a})}_{0} + 1 \Big)\,dx =$$

$$= \int_0^{+\infty} x e^{-x}\,dx = \lim_{b \to +\infty} \left[-e^{-x}(1+x)\right]_0^b =$$

$$= \lim_{b \to +\infty} \Big(-e^{-b}(1+b) - \big(-\underbrace{e^{0}(1+0)}_{1}\big) \Big) =$$

$$= \underbrace{\lim_{b \to +\infty} \left(-e^{-b}\right)}_{0} - \underbrace{\lim_{b \to +\infty} \left(b e^{-b}\right)}_{0} + 1 = 1$$

4. If the probability density of X is given by

$$f(x) = \begin{cases} 1+x & ,\text{for } -1 < x \leq 0 \\ 1-x & ,\text{for } 0 < x < 1 \\ 0 & ,\text{elsewhere} \end{cases}$$

and $U = X$ and $V = X^2$, show that

(a) $cov(U, V) = 0$;

(b) U and V are dependent.

Exercise 4 a)

a)

$$E(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \int_{-1}^{0} x(1+x)\, dx + \int_{0}^{1} x(1-x)\, dx =$$

$$= \int_{-1}^{0} x\, dx + \int_{-1}^{0} x^2\, dx + \int_{0}^{1} x\, dx + \int_{0}^{1} -x^2\, dx =$$

$$= \left[\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^3}{3}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} - \left[\frac{x^3}{3}\right]_{0}^{1} =$$

$$= 0 - \frac{(-1)^2}{2} + 0 - \frac{(-1)^3}{3} + \frac{1}{2} - 0 - \frac{1}{3} - 0 =$$

$$= -\frac{1}{2} + \frac{1}{3} + \frac{1}{2} - \frac{1}{3} = 0$$

$$E(X^3) = \int_{-\infty}^{+\infty} x^3 f_X(x)\, dx = \int_{-1}^{0} x^3(1+x)\, dx + \int_{0}^{1} x^3(1-x)\, dx =$$

$$= \int_{-1}^{0} x^3\, dx + \int_{-1}^{0} x^4\, dx + \int_{0}^{1} x^3\, dx + \int_{0}^{1} -x^4\, dx =$$

$$= \left[\frac{x^4}{4}\right]_{-1}^{0} + \left[\frac{x^5}{5}\right]_{-1}^{0} + \left[\frac{x^4}{4}\right]_{0}^{1} - \left[\frac{x^5}{5}\right]_{0}^{1} =$$

$$= 0 - \frac{(-1)^4}{4} + 0 - \frac{(-1)^5}{5} + \frac{1}{4} - 0 - \frac{1}{5} - 0 =$$

$$= -\frac{1}{4} + \frac{1}{5} + \frac{1}{4} - \frac{1}{5} = 0$$

$$\mathrm{cov}(U, V) = \mathrm{cov}(X, X^2) = E(X X^2) - E(X)E(X^2) =$$

$$= \underbrace{E(X^3)}_{0} - \underbrace{\underbrace{E(X)}_{0} E(X^2)}_{} = 0 \text{ , QED}$$

Exercise 4 b)

We need to show that: $\exists (u,v) \in \mathbb{R}^2 : f_{U,V}(u,v) \neq f_U(u) f_V(v)$

Since $U = X$ we have: $f_U(u) = f_X(u) = \begin{cases} 1+u & (-1 < u < 0) \\ 1-u & (0 < u < 1) \\ 0 & (\text{elsewhere}) \end{cases}$

Now lets find $f_V(v)$ $(0 < v < 1)$:

$$F_V(v) = P(V \leq v) = P(X^2 \leq v) = P(-\sqrt{v} \leq X \leq \sqrt{v}) =$$

$$= F_X(\sqrt{v}) - F_X(-\sqrt{v}) \quad (0 < v < 1)$$

$$f_V(v) = F_X'(\sqrt{v}) \frac{d}{dv}(\sqrt{v}) - F_X'(-\sqrt{v}) \frac{d}{dv}(-\sqrt{v}) =$$

$$= f_X(\sqrt{v}) \frac{1}{2\sqrt{v}} - f_X(-\sqrt{v}) \frac{-1}{2\sqrt{v}} =$$

$$= \frac{f_X(\sqrt{v})}{2\sqrt{v}} + \frac{f_X(-\sqrt{v})}{2\sqrt{v}} = \frac{1-\sqrt{v}}{2\sqrt{v}} + \frac{1-\sqrt{v}}{2\sqrt{v}} = \frac{2-2\sqrt{v}}{2\sqrt{v}} = \frac{1-\sqrt{v}}{\sqrt{v}} \quad (0 < v < 1)$$

Exercise 4 b)

So, $f_V(v) = \begin{cases} \frac{1-\sqrt{v}}{\sqrt{v}} & (0 < v < 1) \\ 0 & (\text{otherwise}) \end{cases}$

Now we only have to find $f_{U,V}(u, v)$.

from this, V and U are obviously dependent!

Since $U = X$ and $V = X^2$ then $V = U^2$ and

$f_{U,V}(u, v) > 0$ only if $v = u^2$ (and $-1 < u < 1$).

So, $f_{U,V}(u,v) = \begin{cases} f_X(u) & (\text{if } v = u^2 \wedge -1 < u < 1) \\ 0 & (\text{otherwise}) \end{cases} = \begin{cases} 1+u & (-1 < u < 0, v = u^2) \\ 1-u & (0 < u < 1, v = u^2) \\ 0 & (\text{elsewhere}) \end{cases}$

Conclusion:

$f_{U,V}(u,v) \neq f_U(u) f_V(v) = \begin{cases} (1+u)(1-\sqrt{v})/\sqrt{v} & (-1 < u < 0, \; 0 < v < 1) \\ (1-u)(1-\sqrt{v})/\sqrt{v} & (0 < u < 1, \; 0 < v < 1) \\ 0 & (\text{elsewhere}) \end{cases}$

Therefore $X \not\perp Y$, QED

6. If the joint probability density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(y+x) & ,\text{for } 0 < x \leq 1,\ 0 < y < 2 \\ 0 & ,\text{elsewhere} \end{cases}$$

Compute $Var(X|Y=y)$.

Exercise 6

$$Var(X|Y=y) = E(X^2|Y=y) - E(X|Y=y)^2 = \frac{3+4y}{6+12y} - \left(\frac{2+3y}{3+6y}\right)^2 =$$

$$= \frac{3+4y}{2(3+6y)} - \frac{4+9y^2+12y}{(3+6y)^2} =$$

$$= \frac{(3+4y)(3+6y) - 2(4+9y^2+12y)}{2(3+6y)^2} =$$

$$= \frac{9+18y+12y+24y^2-8-18y^2-24y}{2(9+36y^2+36y)} = \frac{6y^2+6y+1}{18+72y^2+72y} =$$

This was enough

$$= \frac{6y^2+6y+1}{18(1+4y^2+4y)} = \frac{6y^2+4y+(1+2y)}{18(1+2y)^2} = \frac{1}{18}\left(\frac{6y^2+4y}{(1+2y)^2} + \frac{1+2y}{(1+2y)^2}\right) =$$

$$= \frac{1}{36}\left(2\,\frac{6y^2+4y}{(1+2y)^2} + \frac{2}{1+2y}\right) = \frac{1}{36}\left(\frac{12y^2+8y+2(1+2y)}{(1+2y)^2}\right) =$$

$$= \frac{1}{36}\left(\frac{12y^2+8y+2+4y}{(1+2y)^2}\right) = \frac{1}{36}\left(\frac{12y^2+12y+2}{(1+2y)^2}\right) =$$

$$= \frac{1}{36}\left(\frac{12y^2+12y+3}{(1+2y)^2} - \frac{1}{(1+2y)^2}\right) = \frac{1}{36}\left(\frac{3(4y^2+4y+1)}{(1+2y)^2} - \frac{1}{(2y+1)^2}\right)$$

$$= \frac{1}{36}\left(\frac{3(1+2y)^2}{(1+2y)^2} - \frac{1}{(2y+1)^2}\right) = \frac{1}{36}\left(3 - \frac{1}{(2y+1)^2}\right) \quad (0<y<2) \rightarrow \text{same as solutions}$$

Exercise 6

Auxiliary calculations:

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_0^1 \tfrac{1}{3}(x+y)\,dx =$$

$$= \frac{1}{3}\left[\frac{x^2}{2} + xy\right]_{x=0}^{x=1} = \frac{1}{3}\left(\frac{1}{2}+y\right) \qquad (0<y<2)$$

$$E(X \mid Y=y) = \int_{-\infty}^{+\infty} x\, f_{X|Y=y}(x)\,dx = \int_{-\infty}^{+\infty} x\,\frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \int_0^1 x\,\frac{\frac{1}{3}(y+x)}{\frac{1}{3}\left(\frac{1}{2}+y\right)}\,dx = \int_0^1 x\,\frac{y+x}{y+\frac{1}{2}}\,dx = \int_0^1 \frac{xy+x^2}{y+\frac{1}{2}}\,dx =$$

$$= \frac{1}{y+\frac{1}{2}}\int_0^1 xy + x^2\,dx = \frac{1}{y+\frac{1}{2}}\left[\frac{x^2 y}{2} + \frac{x^3}{3}\right]_{x=0}^{x=1}$$

$$= \frac{\frac{y}{2}+\frac{1}{3}}{\frac{1}{2}+y} = \frac{\frac{2}{3}+y}{1+2y} = \frac{\frac{6}{3}+3y}{3+6y} = \frac{2+3y}{3+6y} \quad (0<y<2) \Rightarrow \text{same as solutions}$$

This was enough

Exercise 6

$$E(X^2|Y=y) = \int_{-\infty}^{+\infty} x^2 f_{X|Y=y}(x)\,dx = \int_{-\infty}^{+\infty} x^2 \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \int_0^1 x^2 \frac{\frac{1}{3}(y+x)}{\frac{1}{3}(\frac{1}{2}+y)}\,dx = \int_0^1 x^2 \frac{y+x}{y+\frac{1}{2}}\,dx = \int_0^1 \frac{x^2y+x^3}{y+\frac{1}{2}}\,dx =$$

$$= \frac{1}{y+\frac{1}{2}} \int_0^1 x^2y+x^3\,dx = \frac{1}{y+\frac{1}{2}} \left[\frac{x^3y}{3} + \frac{x^4}{4}\right]_{x=0}^{x=1}$$

$$= \frac{\frac{y}{3}+\frac{1}{4}}{\frac{1}{2}+y} = \frac{y+\frac{3}{4}}{\frac{3}{2}+3y} = \frac{3+4y}{\frac{12}{2}+12y} = \frac{3+4y}{6+12y} \quad (0<y<2)$$ → same as solutions

This was enough

8. Suppose that $E(Y|X = x) = 9 + x$. If the $E(X) = 11$ and $Var(X) = 290$, what is $cov(Y, X)$?

Exercise 8

Suppose that $E(Y|X = x) = 9 + x$. If the $E(X) = 11$ and $Var(X) = 290$, what is $cov(Y, X)$?

Law of iterated expectations

$$E(Y) = E(E(Y|X)) = E(9 + X) = 9 + E(X) = 9 + 11 = 20$$

$$E(X^2) = Var(X) + E(X)^2 = 290 + 11^2 = 411$$

$$E(XY) = E(E(XY|X)) = E(X\,E(Y|X)) = E(X(9+X)) =$$

$g(X,Y)$

$$= E(9X + X^2) = 9E(X) + E(X^2) = 9 \times 11 + 411 = 510$$

$$cov(X,Y) = E(XY) - E(X)E(Y) =$$

$$= 510 - 11 \times 20 = 510 - 220 = 290$$

Exercise 8

<u>Note (Law of iterated expectations)</u>:

If X and Y are random variables then:

$$E(X) = E(E(X|Y));$$

$$E(Y) = E(E(Y|X))$$

If $Z = g(X, Y)$ then:

$$E(g(X,Y)) = E(Z) = E(E(Z|X)) = E(E(g(X, Y)|X)) =$$

$$E(g(X,Y)) = E(Z) = E(E(Z|Y)) = E(E(g(X,Y|Y)) =$$

9. Let X and Y be two continuous random variables such that the conditional density function of X given $Y = y$ is

$$f_{X|Y=y}(x) = \begin{cases} \frac{1}{2-y}, & y < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and the marginal density of Y is given by

$$f_Y(y) = \begin{cases} 1 - y/2, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}.$$

Compute the expected value of X.

Exercise 9

Let X and Y be two continuous random variables such that the conditional density function of X given $Y = y$ is

$$f_{X|Y=y}(x) = \begin{cases} \frac{1}{2-y}, & y < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

and the marginal density of Y is given by

$$f_Y(y) = \begin{cases} 1 - y/2, & 0 < y < 2 \\ 0, & \text{otherwise} \end{cases}.$$

Compute the expected value of X.

see auxiliary calculation

$$E(X) = \int_0^2 \int_y^2 x \; f_{X,Y}(x,y)\, dx\, dy = \int_0^2 \int_y^2 \frac{x}{2}\, dx\, dy =$$

$$= \frac{1}{2} \int_0^2 \left[\frac{x^2}{2} \right]_{x=y}^{x=2} dy = \frac{1}{2} \int_0^2 2 - \frac{y^2}{2}\, dy =$$

$$= \frac{1}{2} \left[2y - \frac{y^3}{6} \right]_{y=0}^{y=2} = \frac{1}{2}\left(4 - \frac{8}{6} \right) = \frac{1}{2}\left(4 - \frac{4}{3} \right) = \frac{1}{2}\left(\frac{12}{3} - \frac{4}{3} \right) =$$

$$= \frac{8}{6} = \frac{4}{3}$$

Exercise 9

Auxiliary calculation: (to find $f_{X,Y}(x,y)$):

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} \Leftrightarrow \frac{1}{2-y} = \frac{f_{X,Y}(x,y)}{1-\frac{y}{2}} \Leftrightarrow$$

$$\Leftrightarrow \frac{1-\frac{y}{2}}{2-y} = f_{X,Y}(x,y) \Leftrightarrow$$

$$\Leftrightarrow f_{X,Y}(x,y) = \frac{2-y}{2(2-y)} = \frac{1}{2} \quad (y<x<2)$$

Exercise 9

Alternative resolution:

Solution: Firstly, we can compute

$$E(X|Y=y) = \int_{-\infty}^{+\infty} x f_{X|Y=y}(x)dx = \int_y^2 \frac{x}{2-y}dx = 1 + y/2.$$

Secondly, we can use the tower property

$$E(X) = E(E(X|Y)) = E(1 + Y/2) = 1 + E(Y)/2.$$

By definition,

$$E(Y) = \int_{-\infty}^{+\infty} y f_Y(y)dy = \int_0^2 y(1 - y/2)dy = 2/3.$$

Therefore $E(X) = 1 + 1/3 = 4/3$.

11. A company engaged in the trade of various items, whose sales have random behavior. The monthly sales of items A and B, expressed in monetary units, constitute a random vector (X, Y) with joint probability density function given by:

$$f_{X,Y}(x,y) = \frac{1}{2}, \quad 0 < x < 2, 0 < y < x.$$

(a) Compute the means and variaces of X and Y.

(b) Analyze the independence of the two random variables and compute the correlation coefficient.

(c) Find the $E(Y|X=1)$.

(d) Compute the mean and variance of total sales of the two articles.

Exercise 11 a)

$$f_{X,Y}(x,y) = \frac{1}{2}, \quad 0 < x < 2, 0 < y < x.$$

given y we have $y < x < 2$

a)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy = \int_0^x \frac{1}{2}\, dy = \frac{1}{2}\,[y]_{y=0}^{y=x} = \frac{x}{2} \quad (0 < x < 2)$$

$$E(X) = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx = \int_0^2 x \frac{x}{2}\, dx = \frac{1}{2}\int_0^2 x^2\, dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{x=0}^{x=2} =$$

$$= \frac{1}{6}(2^3 - 0) = \frac{8}{6} = \frac{4}{3}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = \int_0^2 x^2 \frac{x}{2}\, dx = \frac{1}{2}\int_0^2 x^3\, dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_{x=0}^{x=2} =$$

$$= \frac{1}{8}(2^4 - 0) = \frac{16}{8} = 2$$

Exercise 11 a)

$$Var(X) = E(X^2) - E(X)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx = \int_y^2 \frac{1}{2}\,dx = \frac{1}{2}\left[x\right]_{x=y}^{x=2} = \frac{2-y}{2} = 1 - \frac{y}{2} \quad (0<y<2)$$

$$E(Y) = \int_{-\infty}^{+\infty} y\, f_Y(y)\,dy = \int_0^2 y\left(1-\frac{y}{2}\right) dy = \int_0^2 y - \frac{y^2}{2}\,dy = \left[\frac{y^2}{2} - \frac{y^3}{6}\right]_{y=0}^{y=2} =$$

$$= \frac{4}{2} - \frac{8}{6} = 2 - \frac{4}{3} = \frac{6}{3} - \frac{4}{3} = \frac{2}{3}$$

$$E(Y^2) = \int_{-\infty}^{+\infty} y^2 f_Y(y)\,dy = \int_0^2 y^2\left(1-\frac{y}{2}\right) dy = \int_0^2 y^2 - \frac{y^3}{2}\,dy = \left[\frac{y^3}{3} - \frac{y^4}{8}\right]_{y=0}^{y=2} =$$

$$= \frac{8}{3} - \frac{16}{8} = \frac{64}{24} - \frac{48}{24} = \frac{16}{24} = \frac{2}{3}$$

Exercise 11 a)

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} = \frac{6}{9} - \frac{4}{9} = \frac{2}{9}$$

Exercise 11 a)

Alternative solution:

$$E(X) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x\, f_{X,Y}(x,y)\, dy\, dx = \int_0^2\int_0^x \frac{x}{2}\, dy\, dx =$$

$$= \frac{1}{2}\int_0^2 x\,[\,y\,]_{y=0}^{y=x}\, dx = \frac{1}{2}\int_0^2 x\,(x-0)\, dx =$$

$$= \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{x=0}^{x=2} = \frac{1}{6}(8-0) = \frac{8}{6} = \frac{4}{3}$$

$$E(X^2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f_{X,Y}(x,y)\, dy\, dx = \int_0^2\int_0^x \frac{x^2}{2}\, dy\, dx =$$

$$= \frac{1}{2}\int_0^2 x^2[\,y\,]_{y=0}^{y=x}\, dx = \frac{1}{2}\int_0^2 x^2(x-0)\, dx =$$

$$= \frac{1}{2}\int_0^2 x^3 dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_{x=0}^{x=2} = \frac{1}{8}(16-0) = 2$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{18}{9} - \frac{16}{9} = \frac{2}{9}$$

Exercise 11 a)

$$E(Y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y\, f_{X,Y}(x,y)\, dy\, dx = \int_0^2\int_0^x \frac{y}{2}\, dy\, dx =$$

$$= \frac{1}{2}\int_0^2 \left[\frac{y^2}{2}\right]_{y=0}^{y=x} dx = \frac{1}{4}\int_0^2 x^2 - 0\ dx =$$

$$= \frac{1}{4}\int_0^2 x^2 dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_{x=0}^{x=2} = \frac{1}{12}(8-0) = \frac{8}{12} = \frac{2}{3}$$

$$E(Y^2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y^2 f_{X,Y}(x,y)\, dy\, dx = \int_0^2\int_0^x \frac{y^2}{2}\, dy\, dx =$$

$$= \frac{1}{2}\int_0^2 \left[\frac{y^3}{3}\right]_{y=0}^{y=x} dx = \frac{1}{6}\int_0^2 x^3 - 0\ dx =$$

$$= \frac{1}{6}\int_0^2 x^3 dx = \frac{1}{6}\left[\frac{x^4}{4}\right]_{x=0}^{x=2} = \frac{1}{24}(16-0) = \frac{16}{24} = \frac{2}{3}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{2}{3} - \left(\frac{2}{3}\right)^2 = \frac{2}{3} - \frac{4}{9} = \frac{6}{9} - \frac{4}{9} = \frac{2}{9}$$

Exercise 11 b)

$$f_X(x)\, f_Y(y) = \frac{x}{2}\left(1-\frac{y}{2}\right) \neq f_{X,Y}(x,y) = \frac{1}{2} \quad (0<x<2,\ 0<y<x)$$

Therefore $X \not\perp Y$.

Alternative solution:

for $0<x<2 \wedge x<y<2$: $f_{X,Y}(x,y) = 0 \neq f_X(x)\, f_Y(y) = \frac{x}{2}\left(1-\frac{y}{2}\right) > 0$

$$\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sigma_X\ \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sqrt{Var(X)\ Var(Y)}} = \frac{1-\frac{4}{3}\cdot\frac{2}{3}}{\sqrt{\frac{2}{9}\times\frac{2}{9}}} = \frac{1-\frac{8}{9}}{\frac{2}{9}} = \frac{\overset{\operatorname{cov}(X,Y)}{\frac{1}{9}}}{\frac{2}{9}} = \frac{1}{2}$$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_{X,Y}(x,y)\, dy\, dx = \int_0^2\int_0^x \frac{xy}{2}\, dy\, dx =$$

$$= \frac{1}{2}\int_0^2 x \int_0^x y\, dy\, dx = \frac{1}{2}\int_0^2 \frac{x}{2}\left[y^2\right]_{y=0}^{y=x} dx = \frac{1}{4}\int_0^2 x(x^2-0)\, dx$$

$$= \frac{1}{4}\int_0^2 x^3\, dx = \frac{1}{4}\left[\frac{x^4}{4}\right]_{x=0}^{x=2} = \frac{1}{16}(16-0) = 1$$

Exercise 11 c)

$$E(Y \mid X=1) = \int_{-\infty}^{+\infty} y \, f_{Y|X=1}(y)\, dy = \int_0^1 y \frac{f_{X,Y}(1,y)}{f_X(1)}\, dy =$$

$$= \int_0^1 y \frac{\frac{1}{2}}{\frac{1}{2}}\, dy = \int_0^1 y\, dy = \left[\frac{y^2}{2}\right]_{y=0}^{y=1} = \frac{1}{2} - 0 = \frac{1}{2}$$

Exercise 11 d)

d)

$X + Y \equiv$ Total sales

$E(X+Y) = E(X) + E(Y) = \frac{4}{3} + \frac{2}{3} = \frac{6}{3} = 2$

$Var(X+Y) = Var(X) + Var(Y) + 2\,cov(X,Y) =$

$= \frac{2}{9} + \frac{2}{9} + 2 \times \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$

Thanks!

Questions?