Lecture 8: Brownian Motion and Options

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Continuous time

- As we assume continuous trading: need to consider continuous time, instead of discrete time
- **Diffusion models** are a standard way to represent random variables in continuous time
- The ideas are analogous to discrete-time stochastic processes
- The basic building block of a diffusion model is a **Brownian motion** (or **Wiener process**), which is a real-valued continuous-time stochastic process

Brownian motion

- Brownian motion is the **random movement** of microscopic particles suspended in a fluid, caused by constant collisions with the fluid molecules.
- It is a classic example of a continuous random walk.
- **Robert Brown** (1827): A Scottish botanist first observed this phenomenon while studying pollen grains in water under a microscope. He noted their erratic movement but could not explain why it happened
- **Albert Einstein** (1905): Provided a theoretical explanation, proving that Brownian motion was due to the random collisions of molecules in a fluid
- **Norbert Wiener** (1923): Developed the mathematical theory of continuous random walks, leading to the Wiener process, a key part of modern stochastic processes

Applications

- Physics and Chemistry:
  - Modeling Particle Motion: Brownian motion provides a model for understanding the random motion of small particles suspended in a fluid, like pollen grains in water or dust particles in air
  - Diffusion: It's crucial for understanding diffusion processes, where particles move from areas of high concentration to low concentration
  - Micromanipulation of DNA: Brownian motion is used in techniques to manipulate DNA molecules
- Finance:
  - Stock Market Modeling: Geometric Brownian motion, a variation of Brownian motion, is used to model the fluctuations of stock prices and other financial assets
  - Options Pricing: The assumption that asset prices follow Brownian motion is essential to options pricing models
- Biology and Medicine: Movement of bacteria, cellular transport
- Computer Science: Randomized algorithms, Monte Carlo methods

Brownian motion

- A **Brownian motion** is the natural generalization of a **random walk** in discrete time
- Can think of a **random walk** as modelling a person's erratic path when intoxicated in discrete time:

$$z_t - z_{t-1} = \varepsilon_t$$

$$\varepsilon_t \sim N(0,1),\ \ E(\varepsilon_t \varepsilon_s) = 0,\ \ s \neq t$$

- A **Brownian motion** z_t:

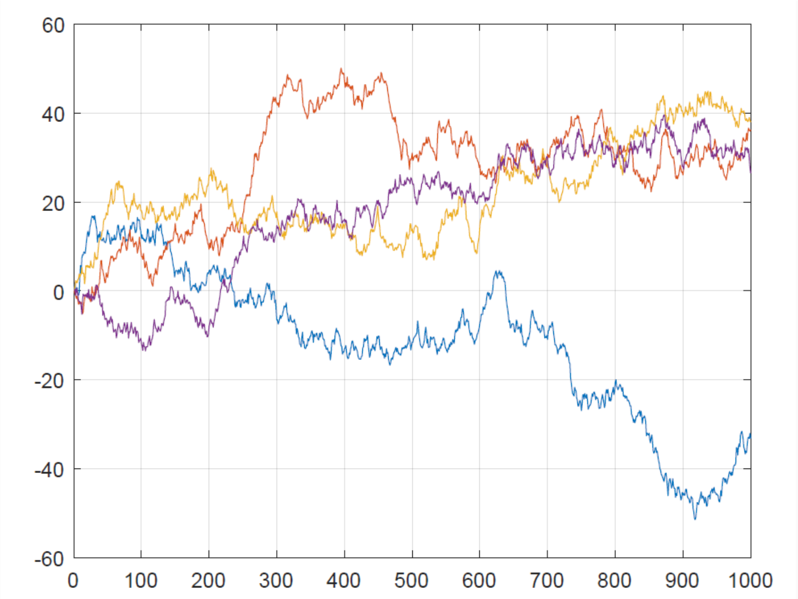
$$z_{t+\Delta} - z_t \sim N(0, \Delta)$$

  i.e. mean zero and variance Δ

As $E(\varepsilon_t \varepsilon_s) = 0$ in discrete time, increments to z for nonoverlapping intervals are also independent

$$cov(z_{t+\Delta} - z_t, z_{s+\Delta} - z_s) = 0$$

Example Brownian motion



Brownian motion

$$dz_t \equiv z_{t+dt} - z_t \sim N(0, dt)$$

- That is, the change in z_t over a small time interval dt, follows a normal distribution with:
  - Mean: 0
  - Variance: dt
  - Independent increments: The increments dz_t over non-overlapping time intervals are independent.

Brownian motion

- The variance of a random walk scales with time

$$var\left(z_{t+k}-z_t\right)=var\left(\varepsilon_{t+1}+...+\varepsilon_{t+k}\right)=kvar\left(z_{t+1}-z_t\right)$$

- And the variance of a Brownian motion scales with time too

$$var(z_{t+k\Delta}-z_t)=kvar(z_{t+\Delta}-z_t)$$

- The standard deviation is the "typical size" of a movement in a normally distributed random variable
- The "typical size" of $z_{t+\Delta}-z_t$ in time interval Δ is $\sqrt[2]{\Delta}$
- This means that $\dfrac{z_{t+\Delta}-z_t}{\Delta}$ has "typical size" $1/\sqrt[2]{\Delta}$
- Thus, the sample path of z_t is continuous but is not differentiable: moves infinitely fast (up and down)

Brownian motion

- **Definition:** Differential dz_t is the forward difference

$$dz_t = \lim_{\Delta \searrow 0} (z_{t+\Delta} - z_t)$$

- Can be represented as an integral

$$z_t = z_0 + \int_0^t dz_t$$

- **Define** dt as the smallest positive real number such that $dt^\alpha = 0$ if $\alpha > 1$
- Properties of dz:

$$\begin{aligned} E_t(dz_t) &= 0 \\ E_t(dz_t dt) &= dt E_t(dz_t) = 0, \text{ } dt \text{ is a constant} \end{aligned}$$

Brownian motion

- Properties of dz:

$$\begin{aligned} dt &= var\left(dz_t\right) = E_t\left[z_{t+\Delta} - z_t - E_t\left(z_{t+\Delta} - z_t\right)\right]^2 \\ &= E_t\left(z_{t+\Delta} - z_t\right)^2 - E_t\left[E_t\left(z_{t+\Delta} - z_t\right)\right]^2 \\ &= E_t\left(z_{t+\Delta} - z_t\right)^2 \equiv E_t\left(dz_t^2\right) \end{aligned}$$

  i.e. the expected value of the squared random variable is the same as the variance.

- **Observation**: notation $dz_t^2 \equiv \left(dz_t\right)^2$

Brownian motion

- Additional properties of dz:

$$var(dz_t^2) = E\left(dz_t^4\right) - E^2\left(dz_t^2\right) = 3dt^2 - dt^2 = 0$$

$$\text{fourth central moment of a normal is } 3\sigma^2 \text{ and } dt^2 \text{ is } 0$$

$$E_t\left(dz_t dt\right)^2 = dt^2 E_t\left(dz_t^2\right) = 0$$

$$var\left(dz_t dt\right) = E_t\left(dz_t dt\right)^2 - E^2\left(dz_t dt\right) = 0$$

$$dz_t^2 = dt, \text{ because the variance of } dz_t^2 \text{ is zero and } E_t\left(dz_t^2\right) = dt$$

$$dz_t dt = 0, \text{ because the variance of } dz_t dt \text{ is zero and } E_t\left(dz_t dt\right) = 0$$

Stochastic differential equation (diffusion)

- Can construct more complicated time-series processes by adding drift, $\mu(\cdot)$, and volatility, $\sigma(\cdot)$, terms to dz_t,

$$dx_t = \mu(t, x_t)\, dt + \sigma(t, x_t)\, dz_t$$

as a short-cut to express

$$x_t = x_0 + \int_0^t \mu(s, x_s)\, ds + \int_0^t \sigma(s, x_s)\, dz_s$$

- Some examples:
  - **Random walk with drift**

$$dx_t = \mu dt + \sigma dz_t, \text{continuous time}$$

$$x_{t+1} - x_t = \mu + \sigma\varepsilon_{t+1}, \text{discrete time}$$

-
  - **Geometric Brownian motion with drift**

$$dx_t = x_t \mu dt + x_t \sigma dz_t$$

Diffusion model

- From the standard Brownian motion case, we already know that $dz_t \sim N(0, dt)$. Since multiplying a normal variable by σ scales its mean and variance, we get

$$\sigma dz_t \sim N(0, \sigma^2 dt)$$

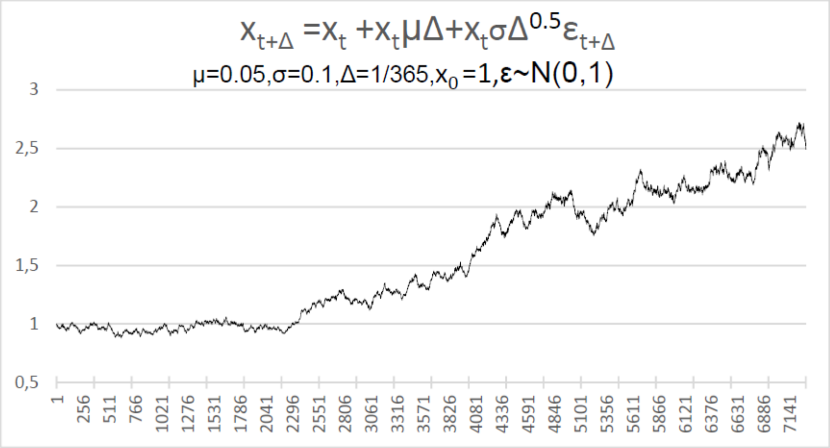
- Adding the drift term μdt gives:

$$\mu dt + \sigma dz_t = dx_t \sim N(\mu dt, \sigma^2 dt)$$

- Any stochastic process (without jumps) can be approximated by a diffusion.

Geometric Brownian motion

Can simulate a diffusion process by approximating it with a small time interval,



Price of stock

- Let P_t be the price of a generic stock at any moment in time that pays dividends at the rate $D_t dt$

The instantaneous return is

$$\frac{dP_t}{P_t} + \frac{D_t}{P_t} dt$$

Let the price be a geometric Brownian motion

$$\frac{dP_t}{P_t} = \mu_p dt + \sigma_p dz_t$$

The risk-free rate can be thought as the return on an asset that does not pay dividend and has the price

$$\frac{dP_t}{P_t} = r_t^f dt$$

Ito's Lemma

- Suppose we have a diffusion representation for one variable, say

$$dx_t = \mu(\cdot)\, dt + \sigma(\cdot)\, dz_t$$

- Define a new variable in terms of the old one,

$$y_t = f(x_t)$$

- What is the diffusion representation for y_t. **Ito's lemma** tells you how to get it
- Use a second-order Taylor expansion, keep terms dz, dt , and $dz^2 = dt$, but terms $dt \times dz$, dt^2, and higher go to zero

Ito's Lemma

- Start with the second order Taylor expansion

$$dy = \frac{df}{dx}dx + \frac{1}{2}\frac{d^2 f}{dx^2}dx^2$$

- Expanding the second term

$$dx^2 = [\mu dt + \sigma dz_t]^2 = \mu^2 dt^2 + \sigma^2 dz_t^2 + 2\mu\sigma dz_t dt = \sigma^2 dt$$

- Substituting for dx^2 and dx

$$\begin{aligned} dy &= \frac{df}{dx}[\mu dt + \sigma dz_t] + \frac{1}{2}\frac{d^2 f}{dx^2}\sigma^2 dt \\ &= \left(\frac{df}{dx}\mu + \frac{1}{2}\frac{d^2 f}{dx^2}\sigma^2\right) dt + \frac{df}{dx}\sigma dz_t \end{aligned}$$

Pricing

- The utility function in continuous time is

$$E_0 \int_0^{\infty} e^{-\delta t} u\left(c_t\right) dt$$

- Let P_t be the price of an asset that pays dividends D_t
- The price must satisfy

$$P_t e^{-\delta t} u'\left(c_t\right) = E_t \int_{s=0}^{\infty} D_{t+s} e^{-\delta(t+s)} u'\left(c_{t+s}\right) ds$$

In discrete time we have:

$$P_t = E_t \sum_{s=0}^{\infty} D_{t+s} \left[\frac{\beta^s u'\left(c_{t+s}\right)}{u'\left(c_t\right)}\right]$$

Pricing

- Define $\Lambda_t \equiv e^{-\delta t} u'(c_t)$ as the discount factor in continuous time. It follows that

$$P_t \Lambda_t = E_t \int_{s=0}^{\Delta} D_{t+s} \Lambda_{t+s} ds + E_t \int_{s=\Delta}^{\infty} D_{t+s} \Lambda_{t+s} ds$$

or

$$P_t \Lambda_t = E_t \int_{s=0}^{\Delta} D_{t+s} \Lambda_{t+s} ds + E_t \left[P_{t+\Delta} \Lambda_{t+\Delta} \right]$$

- For small Δ the integral above can be approximated by $D_t \Lambda_t \Delta$

$$P_t \Lambda_t \approx D_t \Lambda_t \Delta + E_t \left[P_{t+\Delta} \Lambda_{t+\Delta} \right]$$

or

$$0 \approx D_t \Lambda_t \Delta + E_t \left[P_{t+\Delta} \Lambda_{t+\Delta} - \Lambda_t P_t \right]$$

- For $\Delta \longrightarrow dt$

$$0 = D_t \Lambda_t dt + E_t \left[d \left(\Lambda_t P_t \right) \right]$$

Pricing

- Define the function

$$f\left(\Lambda_t P_t\right) = \Lambda_t P_t$$

where

$$d\Lambda_t = \mu_\Lambda dt + \sigma_\Lambda dz_t \text{ and } dP_t = \mu_P dt + \sigma_P dz_t$$

Taylor expansion of $d\left(\Lambda_t P_t\right)$

$$\begin{aligned} d\left(\Lambda_t P_t\right) &= \frac{\partial f}{\partial \Lambda_t} d\Lambda_t + \frac{\partial f}{\partial P_t} dP_t + \frac{1}{2}\frac{\partial^2 f}{\partial \Lambda_t^2}\left(d\Lambda_t\right)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial P_t^2}\left(dP_t\right)^2 + \\ & \frac{1}{2}\frac{\partial^2 f}{\partial P_t \partial \Lambda_t} dP_t d\Lambda_t + \frac{1}{2}\frac{\partial^2 f}{\partial \Lambda_t \partial P_t} d\Lambda_t dP_t \\ & + \text{higher order terms} \end{aligned}$$

Since higher order terms $= 0$, and replacing the derivatives $\frac{\partial^2 f}{\partial \Lambda_t^2} = \frac{\partial^2 f}{\partial P_t^2} = 0$

$$d\left(\Lambda_t P_t\right) = \Lambda_t dP_t + P_t d\Lambda_t + d\Lambda_t dP_t$$

Pricing

- Replacing $d\Lambda_t P_t$ in the pricing equation

$$0 = D_t \Lambda_t dt + E_t \left[d \left(\Lambda_t P_t \right) \right]$$

and dividing by $\Lambda_t P_t$ get

$$0 = \frac{D_t}{P_t} dt + E_t \left[\frac{dP_t}{P_t} + \frac{d\Lambda_t}{\Lambda_t} + \frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} \right]$$

or

$$\frac{D_t}{P_t} dt + E_t \left[\frac{dP_t}{P_t} \right] = -E_t \left[\frac{d\Lambda_t}{\Lambda_t} + \frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} \right]$$

For the risk free rate:

$$D_t = 0, \frac{dP_t}{P_t} = r_t^f dt$$

implying

$$\frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} = 0, \text{ and } r_t^f dt = -E_t \left[\frac{d\Lambda_t}{\Lambda_t} \right]$$

Pricing

- Replacing

$$r_t^f dt = -E_t \left[\frac{d\Lambda_t}{\Lambda_t} \right]$$

in

$$\frac{D_t}{P_t} dt + E_t \left[\frac{dP_t}{P_t} \right] = -E_t \left[\frac{d\Lambda_t}{\Lambda_t} + \frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} \right]$$

- get:

$$\frac{D_t}{P_t} dt + E_t \left[\frac{dP_t}{P_t} \right] = r_t^f dt - E_t \left[\frac{d\Lambda_t}{\Lambda_t} \frac{dP_t}{P_t} \right]$$

which is the equivalent in discrete time to

$$E_t R_{t+1} = R_{t+1}^f - R_{t+1}^f cov_t \left(m_{t+1}, R_{t+1} \right)$$

Pricing

- The Black–Scholes formula provides the price of an option
- We are going to use the discount factor approach
- The risk free bond price follows the process:

$$\frac{dB_t}{B_t} = rdt$$

where r is the riskless rate

- Let S_t be the price of a stock that pays no dividends (alternatively can think that the dividends are already included in the drift: μ):

$$\frac{dS_t}{S_t} = \mu dt + \sigma dz_t$$

- The stochastic discount factor follows the process:

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \frac{\mu - r}{\sigma} dz_t$$

where $\frac{\mu-r}{\sigma}$ is the Sharpe ratio

Pricing

- We established that $\frac{d\Lambda_t}{\Lambda_t}$ must satisfy the pricing condition

$$E_t\left[\frac{dS_t}{S_t}\right] = -E_t\left[\frac{d\Lambda_t}{\Lambda_t} + \frac{d\Lambda_t}{\Lambda_t}\frac{dS_t}{S_t}\right]$$

- Thus, for $\frac{d\Lambda_t}{\Lambda_t}$ to be a stochastic discount factor must satisfy

$$-rdt = E_t\left[\frac{d\Lambda_t}{\Lambda_t}\right], \text{ for the bond and}$$

$$E_t\left[\frac{dS_t}{S_t}\right] - rdt = -E_t\left[\frac{d\left(\Lambda_t\right)}{\Lambda_t}\frac{dS_t}{S_t}\right], \text{ for the stock}$$

Exercise: Check that these 2 conditions are satisfied. Remember $E_t\left(dz_t\right) = 0$, $dz_t^2 = dt$, $dz_t dt = 0$ and $dt^\alpha = 0$, if $\alpha > 1$

Black-Scholes formula

- To find the value of

$$
\begin{aligned}
C_0 &= E_0 \frac{\Lambda_T}{\Lambda_0} \max(S_T - X, 0) \\
&= \int_0^{\infty} \frac{\Lambda_T}{\Lambda_0} \max(S_T - X, 0) df\left(\Lambda_T, S_T\right)
\end{aligned}
$$

- Need to find the values Λ_T $\left(\Lambda_t \equiv e^{-\delta t} u'\left(c_t\right) \text{ for example }\right)$ and S_T
- Need to solve the stochastic differential equation for Λ_t and S_t:

A little Math

$$
\begin{aligned}
d \ln S_t &= \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} dS_t^2 \\
&= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz_t
\end{aligned}
$$

Black-Scholes formula

- Integrating

$$d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz_t$$

from 0 to T gives

$$\int_0^T d \ln S_t = \left(\mu - \frac{1}{2}\sigma^2\right) \int_0^T dt + \sigma \int_0^T dz_t$$

$$\ln S_T = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma \left(z_T - z_0\right)$$

where $z_T - z_0$ is a normally distributed random variable with **mean** zero and **variance** T.

- Thus, $\ln S_T$ is conditionally (on the information at date 0) normal with **mean** $\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T$ and **variance** $\sigma^2 T$.

Black-Scholes formula

- The solutions can be written as

$$\ln S_T(\varepsilon) = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma \sqrt[2]{T}\varepsilon$$

$$\ln \Lambda_T(\varepsilon) = \ln \Lambda_0 - \left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right) T - \frac{\mu - r}{\sigma}\sqrt[2]{T}\varepsilon$$

where

$$\varepsilon = \frac{z_T - z_0}{\sqrt[2]{T}} \sim N(0,1)$$

Recall

$$\frac{d\Lambda_t}{\Lambda_t} = -rdt - \frac{\mu - r}{\sigma} dz_t$$

Black-Scholes formula

- Now we can do the integral:

$$
\begin{aligned}
C_0 &= \int_0^\infty \frac{\Lambda_T}{\Lambda_0} \max(S_T - X, 0) df\left(\Lambda_T, S_T\right) \\
&= \int_{S_T = X}^\infty \frac{\Lambda_T}{\Lambda_0} (S_T - X) df\left(\Lambda_T, S_T\right) \\
&= \int_{\underline{\varepsilon}}^\infty \frac{\Lambda_T\left(\varepsilon\right)}{\Lambda_0} \left(S_T\left(\varepsilon\right) - X\right) f(\varepsilon) d\varepsilon
\end{aligned}
$$

  where f is the density of ε and $\underline{\varepsilon}$ is defined as $S_T\left(\underline{\varepsilon}\right) = X$

- We know the joint distribution of the terminal stock price S_T and discount factor Λ_T on the right hand side, so we have all the information we need to calculate this integral.

Black-Scholes formula

Start by breaking up the integral into two terms

$$C_0 = \int_{\underline{\varepsilon}}^{\infty} \frac{\Lambda_T(\varepsilon)}{\Lambda_0} S_T(\varepsilon) f(\varepsilon) d\varepsilon - X \int_{\underline{\varepsilon}}^{\infty} \frac{\Lambda_T(\varepsilon)}{\Lambda_0} f(\varepsilon) d\varepsilon$$

use

$$\frac{S_T}{S_0} = e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\varepsilon}$$

$$\frac{\Lambda_T}{\Lambda_0} = e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon}$$

$$\begin{aligned} C_0 &= S_0 \int_{\underline{\varepsilon}}^{\infty} e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon} e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon \\ &\quad - X \int_{\underline{\varepsilon}}^{\infty} e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon \end{aligned}$$

Black-Scholes formula

- or

$$C_0 = S_0 \int_{\underline{\varepsilon}}^{\infty} e^{\left(\mu - r - \frac{1}{2}\left(\sigma^2 + \left(\frac{\mu - r}{\sigma}\right)^2\right)\right)T + \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon$$
$$-X \int_{\underline{\varepsilon}}^{\infty} e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon} f(\varepsilon) d\varepsilon$$

Now we replace the formula for $f(\varepsilon)$

$$f(\varepsilon) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\varepsilon^2}$$

$$C_0 = \frac{S_0}{\sqrt{2\pi}} \int_{\underline{\varepsilon}}^{\infty} e^{\left[\mu - r - \frac{1}{2}\left(\sigma^2 + \left(\frac{\mu - r}{\sigma}\right)^2\right)\right]T + \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T}\varepsilon - \frac{1}{2}\varepsilon^2} d\varepsilon$$
$$-\frac{X}{\sqrt{2\pi}} \int_{\underline{\varepsilon}}^{\infty} e^{-\left(r + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2\right)T - \frac{\mu - r}{\sigma}\sqrt{T}\varepsilon - \frac{1}{2}\varepsilon^2} d\varepsilon$$

Black-Scholes formula

- or

$$C_0 = \frac{S_0}{\sqrt{2\pi}} \int_{\underline{\varepsilon}}^{\infty} e^{-\frac{1}{2}\left(\varepsilon - \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T}\right)^2} d\varepsilon$$
$$- \frac{X}{\sqrt{2\pi}} e^{-rT} \int_{\underline{\varepsilon}}^{\infty} e^{-\frac{1}{2}\left(\varepsilon + \frac{\mu - r}{\sigma}\sqrt{T}\right)^2} d\varepsilon$$

- Notice that the integrals have the form of a normal distribution with nonzero mean and variance 1.
- **Recall:** $x \sim N\left(\widetilde{\mu}, \widetilde{\sigma}^2\right)$ if

$$f(x) = \frac{1}{\sqrt{2\pi}\widetilde{\sigma}} e^{-\frac{1}{2}\frac{(x-\widetilde{\mu})^2}{\widetilde{\sigma}^2}}$$

Black-Scholes formula

- Now we compute the $\underline{\varepsilon}$

$$\ln X = \ln S_T = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma\sqrt{T}\underline{\varepsilon}$$

implies

$$\underline{\varepsilon} = \frac{\ln X - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$

- The integrals can be expressed using the cumulative standard normal, Φ

$$\Phi\left(a - \mu\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{(x-\mu)^2}{2}} dx$$

- where $\Phi\left(\cdot\right)$ is the area under the left tail of the standard normal distribution.

Black-Scholes formula

- because Φ is symmetric around zero

$$\Phi\left(a-\mu\right)=1-\Phi\left(\mu-a\right)$$

$$\Phi\left(\mu-a\right)=\frac{1}{\sqrt{2\pi}}\int_{a}^{\infty}e^{-\frac{(x-\mu)^2}{2}}dx$$

Black-Scholes formula

- Substituting in

$$
\begin{aligned}
C_0 &= \frac{S_0}{\sqrt{2\pi}} \int_{\underline{\varepsilon}}^{\infty} e^{-\frac{1}{2}\left(\varepsilon-\left(\sigma-\frac{\mu-r}{\sigma}\right)\sqrt{T}\right)^2} d\varepsilon \\
&\quad - \frac{X}{\sqrt{2\pi}} e^{-rT} \int_{\underline{\varepsilon}}^{\infty} e^{-\frac{1}{2}\left(\varepsilon+\frac{\mu-r}{\sigma}\sqrt{T}\right)^2} d\varepsilon
\end{aligned}
$$

$$
\begin{aligned}
C_0 &= S_0 \Phi\left(-\frac{\ln X - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} + \left(\sigma - \frac{\mu - r}{\sigma}\right)\sqrt{T}\right) \\
&\quad - X e^{-rT} \Phi\left(-\frac{\ln X - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} - \frac{\mu - r}{\sigma}\sqrt{T}\right)
\end{aligned}
$$

- Simplifying, we get the Black-Scholes formula

$$
C_0 = S_0 \Phi\left(\frac{\ln\frac{S_0}{X} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right) - X e^{-rT} \Phi\left(\frac{\ln\frac{S_0}{X} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right)
$$

Black-Scholes formula

- We repeat the formula again here:

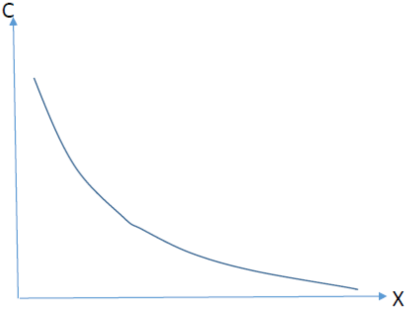
$$C_0 = S_0 \Phi \left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{1}{2}\sigma^2 \right) T}{\sigma \sqrt{T}} \right) - X e^{-rT} \Phi \left(\frac{\ln \frac{S_0}{X} + \left(r - \frac{1}{2}\sigma^2 \right) T}{\sigma \sqrt{T}} \right)$$

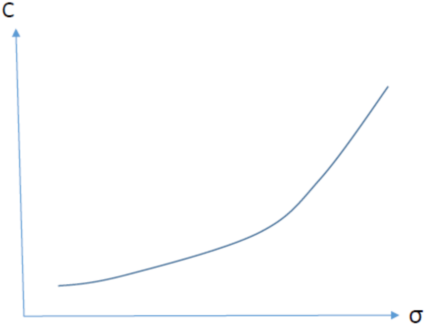
- The price is a function:
  - S_0 (stock price)
  - r (risk free rate)
  - X (strike price)
  - T (time to expiration date)
  - σ (volatility of the underlying stock)

Black-Scholes formula

$$C_0 = S_0 \Phi\left(\frac{\ln\frac{S_0}{X} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right) - Xe^{-rT}\Phi\left(\frac{\ln\frac{S_0}{X} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}\right)$$

- This formula is useful to assess how the price of the option changes when the variables in the r.h.s. of the equation change





Black-Scholes formula

- The price is a monotonic increasing function of the σ
- This formula is often used to solve for σ (once C_0 is known). The σ is the **implied volatility**
- Typically options are quoted in units of sigma

Black-Scholes formula

Exercise:
Determine the price of an European call option with $S_0 = 50$ euros, $r = 4\%$ (annual), $X = 48$ euros, $T = 60$ days and $\sigma = 30\%$ (annual). What is the price of an European put option on the same stock, with the same exercise price and time to maturity?

$$\frac{\ln \frac{S_0}{X} + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} = \frac{\ln \frac{50}{48} + \left(0.04 + \frac{1}{2}\left(0.3\right)^2\right) \frac{60}{365}}{0.3\sqrt{\frac{60}{365}}} = 0.450\,49$$

$$\frac{\ln \frac{S_0}{X} + \left(r - \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}} = \frac{\ln \frac{50}{48} + \left(0.04 - \frac{1}{2}\left(0.3\right)^2\right) \frac{60}{365}}{0.3\sqrt{\frac{60}{365}}} = 0.328\,86$$

$$\Phi\left(0.450\,49\right) = 0.67382$$

Black-Scholes formula

In Excel the command to get the cumulative normal is "=NORM.S.DIST(0,45049;TRUE)"

$$\Phi\left(0.328\,86\right) = 0.62886$$

$$C_0 = 50\left(0.67382\right) - 48e^{-0.04\frac{60}{365}}\left(0.62886\right) = 3.7035$$

To compute the put price must use the put-call parity formula

$$C_0 - P_0 = S_0 - \frac{X}{R^f}$$

$$P_0 = C_0 + \frac{X}{R^f} - S_0$$

$$P_0 = 3.7035 + 48e^{-0.04\frac{60}{365}} - 50 = 1.3889$$

The greeks of options

- In options trading, the "Greeks" are financial measures that quantify how an option's price is affected by various factors, helping traders understand and manage risk. The main Greeks are Delta, Gamma, Theta, and Rho
- Delta (Δ): Measures the sensitivity of an option's price to changes in the underlying asset's price. A delta of 0.5 means the option price is expected to change by \$0.5 for every \$1 change in the underlying asset's price
- Gamma (Γ): Measures the rate of change of an option's delta with respect to changes in the underlying asset's price
- Theta (Θ): Measures the rate at which an option's price decays as time passes
- Rho (ρ): Measures the sensitivity of an option's price to changes in the risk-free interest rate. A rho of 0.01 means the option price is expected to change by \$0.01 for every 1% change in the risk-free interest rate

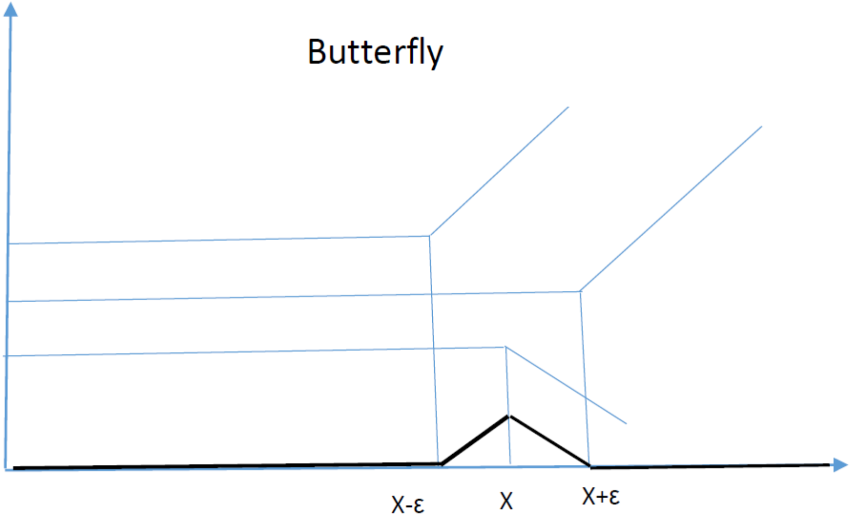
Spanning

- Spanning refers to the ability to replicate the payoff of a financial asset (or any contingent claim) using a portfolio of other available assets
- Given call prices can get discount factors, Arrow-Debreu securities and risk neutral probabilities

Proposition: The second derivative of the call option price with respect to the exercise price is the price of the AD security

Proof: We can construct Arrow-Debreu securities. Consider the strategy of buying 2 call options, one with strike price $X - \varepsilon$ and another with strike price $X + \varepsilon$, and selling 2 call options with strike price X. The payoff of that portfolio (known as butterfly) is

Spanning



Spanning

- The payoff of the contingent claim is the area of the triangle $(0.5\,(2\varepsilon)\,(\varepsilon) = \varepsilon^2)$
- The cost of this portfolio is

$$C\,(X-\varepsilon) - 2C\,(X) + C\,(X+\varepsilon)$$

But this is $\varepsilon^2 C''\,(X)$ (approximately). Recall that $C''\,(x) = \lim_{\varepsilon \longrightarrow 0} \frac{C'(x+\varepsilon)-C'(x)}{\varepsilon}$ and $C'\,(x) = \lim_{\varepsilon \longrightarrow 0} \frac{C(x)-C(x-\varepsilon)}{\varepsilon}$. Thus,

$$C''\,(x) = \lim_{\varepsilon \longrightarrow 0} \frac{\frac{C(x+\varepsilon)-C(x)}{\varepsilon} - \frac{C(x)-C(x-\varepsilon)}{\varepsilon}}{\varepsilon}.$$

Spanning

- Thus, if we buy $\frac{1}{\varepsilon^2}$ of the butterfly we get a payoff of 1 if the $S_T = X$ and a payoff zero for any other value of S_T.

Conclusion: We get an AD security for the state $S_T = X$, and its price is $C''(X)$.

- Once we have AD securities we can price any payoff that is a function of $S_T : G(S_T)$
- The price of a portfolio with payments $G(S_T)$ is

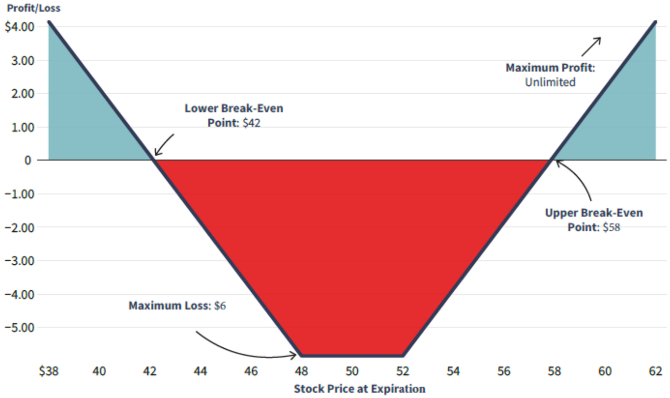
$$P_G = \int_{S_T} C''(X = S_T)\, G(S_T)\, dS_T$$

Spanning

- Since
$$C''(X) = m_{S_T} * 1 * f(S_T = X)$$
where $f(S_T)$ is the probability of S_T
- the discount factor is $m_{S_T} = \frac{C''(X=S_T)}{f(S_T)}$
- Risk neutral probabilities $g(S_T) = (1+r)^T C''(X = S_T)$

$$\begin{aligned} P_G &= \frac{E^g(G(S_T))}{(1+r)^T} = \frac{\int_{S_T} G(S_T)\, g(S_T)\, dS_T}{(1+r)^T} \\ &= \int_{S_T} C''(X = S_T)\, G(S_T)\, dS_T \end{aligned}$$

Options strategies: Long Strangle

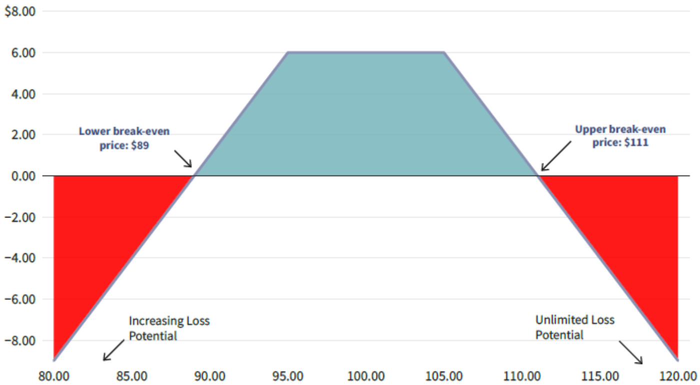


Stock ABC is trading at $50 per share. An investor creates a long strangle this way: Buying a call option with a strike price of $52 and a put option with a strike price of $48, paying a total of $6, both options have the same expiration date. Unlimited profit.

Strangle Strategy

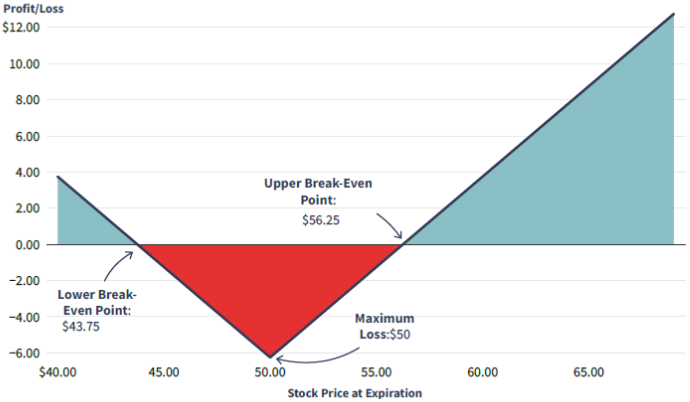
- Strangles are particularly worthwhile during events or market conditions that typically generate significant price volatility
  - **Earnings announcements**: Companies, especially in the tech sectors, often see dramatic price swings after quarterly earnings reports
  - **Merger and acquisition activity**: When companies are rumored to be acquisition targets or involved in major deals
  - **FDA drug approvals**: As we noted above, biotech and pharmaceutical companies often see massive price movements when the FDA makes decisions about their drugs
  - **U.S. Federal Reserve meetings:** Major Fed policy announcements about interest rates or monetary policy can create major market swings
  - **Major product launches:** Companies like Apple often see their shares undergo significant movements around major product announcements

Short Strangle Strategy



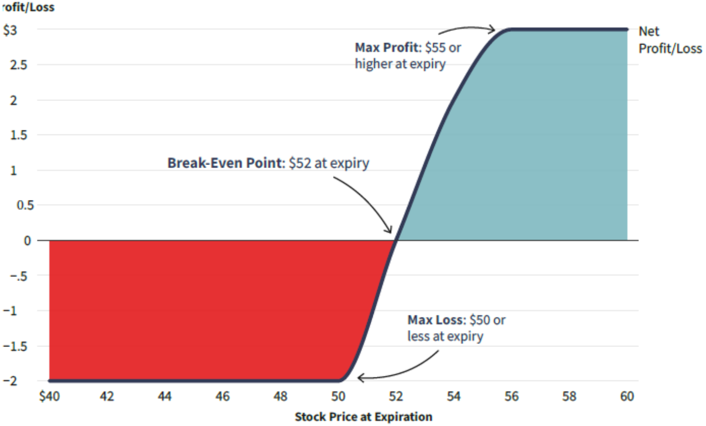
Stock ABC is trading at $100 per share. An investor creates a short strangle this way: Selling a put option with a $95 strike price, receiving a $3 price, selling a call option with a $105 strike price, receiving a $3 price, both options have the same expiration date. Maximum profit $6

Long Straddle Options Strategy



Stock ABC is trading at $50 per share. Buy a call option with a strike price of $50 and a put option with a strike price of $50. The potential profit is unlimited if the stock price moves significantly up or down, while the maximum loss is limited to the total premium paid ($6.25) if the stock price remains at $50 at expiration

Bull Call Spread Options Strategy



An investor, expecting a stock to rise, carries out a bull call spread. They buy a call option with a strike price of $50 for $3 and sell a call option with a strike price of $55 for $1. The net cost of this spread is $2.

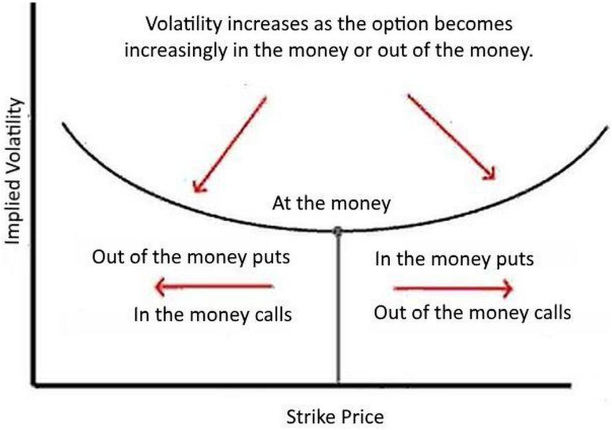
Bear Call Spread Options Strategy

An investor believes a stock, trading at $52, will fall in price. Using a bear put strategy, the investor buys one put option with a strike price of $50 (higher strike) and sells one put option with a strike price of $45 (lower strike)

Data

- Are actual prices equal to the ones predicted by the Black-Scholes formula?
- When options with the same maturity T, same S, but different X, are graphed for implied volatility (IV) the tendency is for that graph to show a **smile**.
- The smile shows that the options that are furthest in or out-of-the-money have the highest implied volatility.
- Options with the lowest IV have strike prices at or near-the-money.
- **But** the Black-Scholes model predicts that the IV curve is flat when plotted against varying strike prices**!**

Data


Volatility increases as the option becomes
increasingly in the money or out of the money.
Implied Volatility
At the money
Out of the money puts
In the money calls
In the money puts
Out of the money calls
Strike Price

Data

- The IV is a key input into option pricing models like Black-Scholes. Higher IV implies higher option price
- This means that calls near-the-money have a lower price than the others
- What does this mean intuitively?
  - Markets are assigning more probability to extreme price moves — in both directions — than the Black-Scholes model would suggest
  - So: out of the money calls are priced higher to reflect the higher chance of big moves (up or down)

Data

- Why does this happen?
  - Fat tails: Real asset returns have more extreme events than a normal distribution predicts
  - Skewed risk perceptions: Traders might fear large drops more than gains
  - Demand/supply imbalances: Investors might hedge tail risks more aggressively, bidding up OTM options.
- Solution: Consider that the underlying asset price follows a distribution with fatter tails, or that the volatility is a stochastic process too!

$$\frac{d\sigma_t}{\sigma_t} = \widetilde{\mu} dt + \widetilde{\sigma} dz_t$$