Lisbon School of Economics & Management
Universidade de Lisboa

Master in Mathematical Finance
Probability Theory and Stochastic Processes
First Semester - 2025/2026

List 1 - Measure and probability

1. Decide if $\mathcal{F}$ is a σ-algebra of Ω, where:
   (a) $\mathcal{F} = \mathcal{P}(\Omega)$, $\Omega = \mathbb{R}^n$.
   (b) $\mathcal{F} = \{\emptyset, \{1,2\}, \{3,4,5,6\}, \Omega\}$, $\Omega = \{1,2,3,4,5,6\}$.
   (c) $\mathcal{F} = \big\{\emptyset, \{0\}, \mathbb{R}^-, \mathbb{R}_0^-, \mathbb{R}^+, \mathbb{R}_0^+, \mathbb{R}\setminus\{0\}, \mathbb{R}\big\}$, $\Omega = \mathbb{R}$.

2. Let $(\Omega, \mathcal{F})$ be a measurable space and $A_1, A_2, \dots \in \mathcal{F}$. Prove:
   (a) $\displaystyle\bigcap_{i=1}^{+\infty} A_i \in \mathcal{F}$
   (b) $A_1 \setminus A_2 \in \mathcal{F}$

3. Let Ω be a finite set with $\#\Omega = n$. Compute $\#\mathcal{P}(\Omega)$. *Hint*: Find a bijection between $\mathcal{P}(\Omega)$ and the space $\{v \in \mathbb{R}^n \colon v_i \in \{0,1\}\}$.

4. Let Ω be an infinite set, i.e. $\#\Omega = +\infty$. Consider de collection of all finite subsets of Ω,
$$\mathcal{C} = \{A \in \mathcal{P}(\Omega) \;:\; \#A < +\infty\}.$$
   Is $\mathcal{C} \cup \Omega$ an algebra? Is it a σ-algebra?

5. Let $(\Omega, \mathcal{F})$ be a measurable space. Consider two disjoint sets $A, B \subset \Omega$ and assume that $A \in \mathcal{F}$. Show that $A \cup B \in \mathcal{F}$ is equivalent to $B \in \mathcal{F}$.

6. Let $\Omega = [-1,1] \subset \mathbb{R}$. Determine if the following collection of sets is a σ-algebra:
$$\mathcal{F} = \{A \in \mathcal{B}(\Omega) \colon x \in A \Rightarrow -x \in A\}.$$

7. (a) Prove that the intersection of σ-algebras is a σ-algebra.
   (b) Determine if the union of σ-algebras is a σ-algebra.

8. Show that
   (a) if $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \mathcal{P}$, then $\sigma(\mathcal{I}_1) \subset \sigma(\mathcal{I}_2)$.
   (b) $\sigma(\sigma(\mathcal{I})) = \sigma(\mathcal{I})$ for any $\mathcal{I} \subset \mathcal{P}$.

9. Consider $\mathcal{B}(\Omega)$ as defined in section 1.2. (Borel sets) of the lecture notes. Prove that:

   (a) $\mathcal{B}(\Omega)$ is an σ-algebra of Ω.

   (b) Any singular set $\{a\}$, with $a \in \mathbb{R}$, is a Borel set.

   (c) Any countable set is a Borel set.

   (d) Any open set is a Borel set. *Hint:* Any open set can be written as a countable union of pairwise disjoint open intervals.

10. Let $(\Omega, \mathcal{F})$ be a measurable space and $A, B \in \mathcal{F} \setminus \{\emptyset, \Omega\}$ such that $A \neq B$. Compute $\#\sigma(\{A, B\})$ if

   (a) $A \cap B \neq \emptyset$ and $A \cup B \neq \Omega$.

   (b) $A \cap B = \emptyset$ and $A \cup B \neq \Omega$.

   (c) $A \cap B \neq \emptyset$ and $A \cup B = \Omega$.

   (d) $A \cap B = \emptyset$ and $A \cup B = \Omega$.

11. Let $\mu\colon \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ be given by $\mu(\emptyset) = 0$, $\mu(\mathbb{R}) = 2$ and $\mu(X) = 1$ if $X \in \mathcal{P}(\mathbb{R}) \setminus \{\emptyset, \mathbb{R}\}$. Determine if μ is σ-subadditive and σ-additive.

12. (Counting measure) Show that the function that counts the number of elements of a set $A \in \mathcal{P}(\Omega)$,

$$\mu(A) = \begin{cases} \#A, & \#A < +\infty \\ +\infty, & \text{o.c.} \end{cases},$$

   is a measure.

13. Let $\mu_n\colon \mathcal{A} \to \bar{\mathbb{R}}$ be a measure and $a_n \geq 0$ for all $n \in \mathbb{N}$. Prove that $\mu\colon \mathcal{A} \to \bar{\mathbb{R}}$ defined by

$$\mu(A) = \sum_{n=1}^{+\infty} a_n \, \mu_n(A), \quad \text{for any } A \in \mathcal{A},$$

   is also a measure. Furthermore, show that if μ_n is a probability measure for all n and $\sum_n a_n = 1$, then μ is also a probability measure.

14. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Show that for any sequence of measurable sets $A_1, A_2, \dots \in \mathcal{F}$ we have

$$\mu\left(\bigcup_{n\geq 1} A_n\right) \leq \sum_{n\geq 1} \mu(A_n).$$

15. Consider two sets each one having full measure. Show that their intersection also has full measure.

16. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $A_1, A_2, \dots \in \mathcal{F}$. Prove that:

   (a) In the definition of measure, the condition $\mu(\emptyset) = 0$ can be replaced by the existence of a set $E \in \mathcal{F}$ with finite measure, $\mu(E) < +\infty$.

   (b) If $A_i \subset A_{i+1}$, then $\mu\left(\bigcup_i A_i\right) = \lim_{i \to +\infty} \mu(A_i)$.

   (c) If $A_{i+1} \subset A_i$ and $\mu(A_1) < +\infty$, then $\mu\left(\bigcap_i A_i\right) = \lim_{i \to +\infty} \mu(A_i)$.

17. Let $\mathcal{A}$ be an algebra of a set Ω. Prove that, $\mathbb{P} : \mathcal{A} \to [0, +\infty]$ is a probability measure if and only if

   (1) $\mathbb{P}(\Omega) = 1$,

   (2) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ for every disjoint sets $A, B \in \mathcal{A}$,

   (3) $\lim_{n \to +\infty} \mathbb{P}(A_n) = 0$ for all $A_1, A_2, \dots \in \mathcal{A}$ such that $A_n \downarrow \emptyset$.

18. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $A_1, A_2, \dots \in \mathcal{F}$, and B the set of points in Ω that belong to an infinite number of A_n's, i.e.

$$B = \bigcap_{n=1}^{+\infty} \bigcup_{k=n}^{+\infty} A_k.$$

   Show that:

   (a) (First Borel-Cantelli lemma) If $\sum_{n=1}^{+\infty} \mathbb{P}(A_n) < +\infty$, then $\mathbb{P}(B) = 0$.

   (b) *(Second Borel-Cantelli lemma) If

$$\sum_{n=1}^{+\infty} \mathbb{P}(A_n) = +\infty \qquad \text{and} \qquad \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mathbb{P}(A_i),$$

   for every $n \in \mathbb{N}$ (i.e. the events are mutually independent), then $\mathbb{P}(B) = 1$.

19. Is $\mathbb{P} : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ defined by

$$\mathbb{P}(A) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \delta_{\frac{1}{n}}(A), \quad A \in \mathcal{P}(\mathbb{R}),$$

   a probability measure on $\mathcal{P}(\mathbb{R})$?

20. Let $a, b \in \mathbb{R}$. Compute the Lebesgue measure of the following Borel sets:

   (a) $[a, b]$.

   (b) $[a, b[$.

   (c) $]-\infty, a[$.

   (d) $]-\infty, a]$.

   (e) $[b, +\infty[$.

   (f) $]b, +\infty[$.