List 2 - Measurable functions

1. Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces and consider a function $f : \Omega_1 \to \Omega_2$. Show that:

   (a) If f is $(\mathcal{F}_1, \mathcal{F}_2)$-measurable, it is also $(\mathcal{F}, \mathcal{F}_2)$-measurable for any σ-algebra $\mathcal{F} \supset \mathcal{F}_1$.

   (b) If f is $(\mathcal{F}_1, \mathcal{F}_2)$-measurable, it is also $(\mathcal{F}_1, \mathcal{F})$-measurable for any σ-algebra $\mathcal{F} \subset \mathcal{F}_2$.

2. Show that:

   (a) $\mathcal{X}_{f^{-1}(A)} = \mathcal{X}_A \circ f$ for any $A \subset \Omega_2$ and $f : \Omega_1 \to \Omega_2$.

   (b) $\mathcal{X}_{A \cap B} = \mathcal{X}_A \mathcal{X}_B$ for $A, B \subset \Omega$.

   (c) $\mathcal{X}_{A \cup B} = \mathcal{X}_A + \mathcal{X}_B - \mathcal{X}_{A \cap B}$ for $A, B \subset \Omega$.

3. Let $f : \Omega_1 \to \Omega_2$ and $g : \Omega_2 \to \Omega_3$ be measurable functions. Prove that $g \circ f$ is also measurable.

4. Show that a function f is measurable iff f^+ and f^- are measurable.
   Hint: $f^+ = \mathcal{X}_A f$ where $A = \{x \in \Omega : f(x) > 0\}$.

5. Let $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable function. Show that:

   (a) $|f|$ is also measurable.

   (b) $f^{-1}(\{a\}) = \{x : f(x) = a\}$ is a measurable set.

6. Let $\mathcal{A} \subset \mathcal{F}$ be σ-algebras. Are the following propositions true? If not, write examples that contradict the statements.

   (a) If a function is $\mathcal{A}$-measurable, then it is also $\mathcal{F}$-measurable.

   (b) If a function is $\mathcal{F}$-measurable, then it is also $\mathcal{A}$-measurable.

7. Consider a simple function φ. Write $|\varphi|$ and determine if it is also simple.

8. Let $([0,1], \mathcal{B}([0,1]), m)$ be the Lebesgue measure space and $f_n(x) = x^n$, $x \in [0,1]$, $n \in \mathbb{N}$. Determine the convergence of f_n.

9. Show that if the limit of a sequence of measurable functions exists, it is also measurable.

10. Let $(\Omega_1, \mathcal{F}_1, \mu)$ be a measure space, $(\Omega_2, \mathcal{F}_2)$ a measurable space, and $f\colon \Omega_1 \to \Omega_2$ a measurable function. Show that:

    (a) The function $\mu \circ f^{-1}$ is a measure on $\mathcal{F}_2$.

    (b) If μ is a probability measure, then $\mu \circ f^{-1}$ is also a probability measure.

11. Let $(\Omega, \mathcal{P}, \delta_a)$ be a measure space where δ_a is the Dirac measure at $a \in \Omega$. If $f\colon \Omega \to \mathbb{R}$ is measurable, what is its induced measure (distribution)?

12. Compute $m \circ f^{-1}$ where $f(x) = 2x$, $x \in \mathbb{R}$, and m is the Lebesgue measure on $\mathbb{R}$.