List 4 - Distributions

1. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a $\mathcal{B}(\mathbb{R})$-measurable function and α is the distribution of a random variable X. Find the distribution of $f \circ X$.

2. Show that $\operatorname{Var}(X) = 0$ iff $\mathbb{P}(X = \mathbb{E}(X)) = 1$.

3. Consider $X_1, \dots, X_n$ integrable random variables. Show that if $\operatorname{Cov}(X_i, X_j) = 0$, $i \neq j$, then

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i).$$

4. Let X be a random variable and $\lambda > 0$. Prove the *Chebyshev's inequalities*:

   (a) For $k \in \mathbb{N}$,

   $$\mathbb{P}(|X| \geq \lambda) \leq \frac{1}{\lambda^k}\mathbb{E}(|X|^k).$$

   When $k = 1$ this is also called *Markov inequality*.

   (b) If $\mathbb{E}(X)$ is finite,

   $$\mathbb{P}\left(|X - \mathbb{E}(X)| \geq \lambda\right) \leq \frac{\operatorname{Var}(X)}{\lambda^2}.$$

5. Find the distribution of a simple function in the form

$$X = \sum_{j=1}^{N} c_j \mathcal{X}_{A_j}$$

   and compute its moments.

6. Determine the distribution functions of the following probability measures on $\mathbb{R}$ (δ_a is the Dirac measure at a and m_A is the Lebesgue measure on A):

   (a) $\mu = \frac{1}{3}\delta_2 + \ \frac{2}{3}\delta_3$

   (b) $\mu = \frac{1}{3}\delta_2 + \ \frac{1}{3}\delta_3 + \frac{1}{3}m_{[0,1]}$

7. Show that for $-\infty \leq a \leq b \leq +\infty$ we have

   (a) $\alpha(\{a\}) = F(a) - F(a^-)$

   (b) $\alpha(]a, b[) = F(b^-) - F(a)$

   (c) $\alpha([a, b[) = F(b^-) - F(a^-)$

   (d) $\alpha([a, b]) = F(b) - F(a^-)$

   (e) $\alpha(]-\infty, b[) = F(b^-)$

   (f) $\alpha([a, +\infty[) = 1 - F(a^-)$

   (g) $\alpha(]a, +\infty[) = 1 - F(a)$

8. Compute the distribution function of the following distributions:

   (a) The Dirac distribution δ_a at $a \in \mathbb{R}$.

   (b) The Bernoulli distribution $p\delta_a + (1-p)\delta_b$ with $0 \leq p \leq 1$ and $a, b \in \mathbb{R}$.

   (c) The uniform distribution on a bounded interval $I \subset \mathbb{R}$

$$m_I(A) = \frac{m(A \cap I)}{m(I)}, \qquad A \in \mathcal{B}(\mathbb{R}),$$

   where m is the Lebesgue measure.

   (d) $\alpha = c_1\delta_a + c_2 m_I$ on $\mathcal{B}(\mathbb{R})$ where $c_1, c_2 \geq 0$ and $c_1 + c_2 = 1$.

   (e) $\alpha = \displaystyle\sum_{n=1}^{+\infty} \frac{1}{2^n}\delta_{-1/n}$.

9. Find the distribution functions of the following discrete distributions $\alpha(A) = P(X \in A)$, $A \in \mathcal{B}(\mathbb{R})$:

   (a) Degenerate (or Dirac or atomic) distribution: $\alpha(A) = \begin{cases} 1, & a \in A \\ 0, & \text{o.c.} \end{cases}$, where $a \in \mathbb{R}$.

   (b) Binomial distribution with $n \in \mathbb{N}$: $\alpha(\{k\}) = C_k^n\, p^k(1-p)^{n-k}, \quad 0 \leq k \leq n.$

   (c) Poisson distribution with $\lambda > 0$: $\alpha(\{k\}) = \frac{\lambda^k}{k!e^\lambda}, \quad k \in \mathbb{N} \cup \{0\}.$
   This describes the distribution of 'rare' events with rate λ.

   (d) Geometric distribution with $0 < p < 1$: $\alpha(\{k\}) = (1-p)^k p, \quad k \in \mathbb{N} \cup \{0\}.$
   This describes the distribution of the number of unsuccessful attempts preceding a success with probability p.

   (e) Negative binomial distribution: $\alpha(\{k\}) = C_k^{n+k-1}(1-p)^k p^n, \quad k \in \mathbb{N} \cup \{0\}.$
   This describes the distribution of the number of accumulated failures before n successes.

10. Find the distribution functions of the following absolutely continuous distributions $\alpha(A) = P(X \in A) = \int_A f(x)\,dx$, $A \in \mathcal{B}(\mathbb{R})$ where f is the density function:

   (a) Uniform distribution on $[a, b]$: $f(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & \text{o.c.} \end{cases}$

   (b) Exponential distribution: $f(x) = e^{-x}, \quad x \geq 0$

   (c) The two-sided exponential distribution: $f(x) = \frac{1}{2}e^{-|x|}, \quad x \in \mathbb{R}$

   (d) The Cauchy distribution: $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \quad x \in \mathbb{R}$

11. Let $X_n \sim \mathrm{U}([0,1])$ and $Y_n = -\frac{1}{\lambda}\log(1 - X_n)$, with $\lambda > 0$. Show that $Y_n \sim \mathrm{Exp}(\lambda)$.