ISEG Lisbon School of Economics & Management Universidade de Lisboa

Master in Mathematical Finance

Probability Theory and Stochastic Processes

First Semester - 2025/2026

List 5 - Independence

1. Prove that:

   (a) If A_1 and A_2 are independent, then A_1^c and A_2 are also independent.

   (b) Any full probability event is independent of any other event. The same for any zero probability event.

   (c) Two disjoint events are independent iff at least one of them has zero probability.

   (d) Consider two events $A_1 \subset A_2$. They are independent iff A_1 has zero probability or A_2 has full probability.

2. Suppose that A and C are independent events as well as B and C with $A \cap B = \emptyset$. Show that $A \cup B$ and C are also independent.

3. Give examples of probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$, and of events A_1 and A_2 such that $\mathbb{P}_1(A_1 \cap A_2) = \mathbb{P}_1(A_1)\,\mathbb{P}_1(A_2)$ but $\mathbb{P}_2(A_1 \cap A_2) \neq \mathbb{P}_2(A_1)\,\mathbb{P}_2(A_2)$. Notice that the definition of independence depends on the probability measure.

4. Show that any random variable is independent of a constant random variable.

5. Let X and Y be independent random variables and f and g $\mathcal{B}$-measurable functions on $\mathbb{R}$. Prove that:

   (a) $f(X)$ and $g(Y)$ are independent.

   (b) $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\,\mathbb{E}(g(Y))$ if $\mathbb{E}(|f(X)|), \mathbb{E}(|g(Y)|) < +\infty$.

6. Suppose that the random variables X and Y have only values in $\{0, 1\}$. Show that if $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$, then X, Y are independent.

7. Construct an example of two uncorrelated random variables that are not independent.

8. Show that if $\mathrm{Var}(X) \neq \mathrm{Var}(Y)$, then $X + Y$ and $X - Y$ are not independent.

9. Prove that:

   (a) If $\mathcal{G} \subset \mathcal{F}_1$ and $\mathcal{F}_1, \mathcal{F}_2$ are independent σ-algebras, then $\mathcal{G}$ and $\mathcal{F}_2$ are also independent.

   (b) Two random variables X, Y are independent iff $\sigma(X)$ and $\sigma(Y)$ are independent.