Master in Mathematical Finance
Probability Theory and Stochastic Processes
First Semester - 2025/2026

List 6 - Conditional expectation

1. Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let X be an integrable random variable. Prove that:

   (a) $\mathbb{E}(X|\Omega) = \mathbb{E}(X)$;

   (b) given $A, B \in \mathcal{F}$, with $\mathbb{P}(B) > 0$, then $\mathbb{E}(\mathcal{X}_A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$;

   (c) if Y is a constant function, then $\mathbb{E}(X|Y) = \mathbb{E}(X)$;

   (d) given $B \in \mathcal{F}$ such that $0 < \mathbb{P}(B) < 1$, then:

      i. $\mathbb{E}(X|\mathcal{X}_B)(\omega) = \begin{cases} \dfrac{1}{\mathbb{P}(B)} \displaystyle\int_B X \, d\mathbb{P}, & \text{if } \omega \in B \\ \dfrac{1}{\mathbb{P}(B^c)} \displaystyle\int_{B^c} X \, d\mathbb{P}, & \text{if } \omega \in B^c \end{cases}$;

      ii. $\mathbb{E}(\mathcal{X}_A|\mathcal{X}_B)(\omega) = \begin{cases} \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}, & \text{if } \omega \in B \\ \frac{\mathbb{P}(A\cap B^c)}{\mathbb{P}(B^c)}, & \text{if } \omega \in B^c \end{cases}$.

2. Let X be an integrable random variable.

   (a) Show that if $0 < \mathbb{P}(B) < 1$ and $\alpha, \beta \in \mathbb{R}$, then

$$\mathbb{E}(X|\alpha\mathcal{X}_B + \beta\mathcal{X}_{B^c}) = \mathbb{E}(X|\mathcal{X}_B).$$

   (b) Given $Y = \alpha\mathcal{X}_{B_1} + \beta\mathcal{X}_{B_2}$ where $B_1 \cap B_2 = \emptyset$ and $\alpha \neq \beta$. Find $\mathbb{E}(X|Y)$.

3. Let $\Omega = \{1, 2, ..., 6\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, $\quad \mathbb{P}(\{x\}) = \begin{cases} \frac{1}{16}, & x = 1, 2 \\ \frac{1}{4}, & x = 3, 4 \\ \frac{3}{16}, & x = 5, 6, \end{cases}$,

$$X(x) = \begin{cases} 2, & x = 1, 2 \\ 8, & x = 3, ..., 6, \end{cases}, \quad \text{and} \quad Y = 4\mathcal{X}_{\{1,2,3\}} + 6\mathcal{X}_{\{4,5,6\}}.$$

   Find $\mathbb{E}(X|Y)$.

4. Prove that, if Y is a discrete random variable, then:

   (a) for any event B on which Y is constant, $\displaystyle\int_B \mathbb{E}(X|Y) \, d\mathbb{P} = \int_B X \, d\mathbb{P}$, and

   (b) $\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X)$.

5. Let $\Omega = [0,1[$, $\mathcal{F} = \mathcal{B}([0,1[)$ and $\mathbb{P} = m$ where m is the Lebesgue measure on $[0,1[$. Consider the random variables $X(x) = x$ and

$$Y(x) = \begin{cases} 2x, & 0 \le x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \le x < 1. \end{cases}$$

   (a) Find $\sigma(Y)$.

   (b) By the knowledge that $\mathbb{E}(X|Y)$ is $\sigma(Y)$-measurable, show that

$$\mathbb{E}(X|Y)(x) = \mathbb{E}(X|Y)(x + 1/2), \quad 0 \le \omega < 1/2.$$

   (c) Reduce the problem of determining $\mathbb{E}(X|Y)$ on $[0,1[$ to finding the solution of

$$\int_A \mathbb{E}(X|Y)\, dm = \frac{1}{2} \int_{A \cup (A+1/2)} X\, dm, \quad A \in \mathcal{B}([0,1/2[),$$

   and compute $\mathbb{E}(X|Y)$.

6. Prove that if $\mathcal{G} = \{\emptyset, \Omega\}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.

7. Prove that, if $B \in \mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|B) = \mathbb{E}(X|B)$.

8. Prove that:

   (a) If $\mathbb{P}(A) = 0$, then $\mathbb{P}(A|\mathcal{G}) = 0$ a.s.

   (b) If $\mathbb{P}(A) = 1$, then $\mathbb{P}(A|\mathcal{G}) = 1$ a.s.

   (c) $0 \le \mathbb{P}(A|\mathcal{G}) \le 1$ a.s. for any $A \in \mathcal{F}$.

   (d) If $A_1, A_2, \dots \in \mathcal{F}$ are pairwise disjoint, then $\mathbb{P}\left(\bigcup_{n=1}^{+\infty} A_n \middle| \mathcal{G}\right) = \sum_{n=1}^{+\infty} \mathbb{P}(A_n|\mathcal{G})$ a.s.

   (e) If $A \in \mathcal{G}$, then $\mathbb{P}(A|\mathcal{G}) = \mathcal{X}_A$ a.s.

9. Let $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. Compute:

   (a) $\sigma(\mathcal{X}_B)$.

   (b) $\mathbb{E}(X|\mathcal{X}_B)$ for any random variable X.

   (c) $\mathbb{P}(A|\mathcal{X}_B)$ where $A \in \mathcal{F}$.

10. Prove that for $A, B \in \mathcal{F}$:

   (a) Assuming that $\mathbb{P}(B) > 0$,

$$A \text{ and } B \text{ are independent events iff } \mathbb{P}(A|B) = \mathbb{P}(A);$$

   (b) $\mathbb{P}(A|B)\,\mathbb{P}(B) = \mathbb{P}(B|A)\,\mathbb{P}(A)$.

(c) For any sequence of pairwise disjoint sets $C_1, C_2, \dots \in \mathcal{F}$ such that $\mathbb{P}\left(\bigcup_{n=1}^{+\infty} C_n\right) = 1$, we have

$$\mathbb{P}(A|B) = \sum_{n=1}^{+\infty} \mathbb{P}(A \cap C_n|B) \qquad \text{and} \qquad \mathbb{P}(A|B) = \sum_{n=1}^{+\infty} \mathbb{P}(A|C_n \cap B)\, \mathbb{P}(C_n|B).$$