Master in Mathematical Finance
Probability Theory and Stochastic Processes
First Semester - 2025/2026

List 7 - Markov chains

1. Prove that the following propositions are equivalent[1]

   (a) P is a stochastic $n \times n$ matrix (i.e. all its coefficients are in $[0,1]$ and their sum over each row is equal to 1).

   (b) For any $v \geq 0$ in $\mathbb{R}^n$ we have $Pv \geq 0$ and $P(1,\dots,1) = (1,\dots,1)$.

   (c) For any probability vector $v \in \mathbb{R}^n$ (i.e. $v \geq 0$ and $\sum_{i=1}^r v_i = 1$), we have that vP is also a probability vector.

2. Show that the product of two stochastic matrices is also a stochastic matrix.

3. (Chapman-Kolmogorov equations) Consider a homogeneous Markov chain with state space S. Prove that:

   (a) $\pi_{i,j}^{(n)} = \sum_{k \in S} \pi_{i,k}\, \pi_{k,j}^{(n-1)}$, for all $n \in \mathbb{N}$ and $i,j \in S$.

   (b) $\pi_{i,j}^{(n+m)} = \sum_{k \in S} \pi_{i,k}^{(n)}\, \pi_{k,j}^{(m)}$, for all $m, n \in \mathbb{N}_0$ and $i,j \in S$.

   Hint: Recall the following facts: $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\,\mathbb{P}(B|C)$ and $\mathbb{P}(A|B) = \sum_n \mathbb{P}(A \cap C_n|B)$, where $C_1, C_2, \dots$ is a sequence of disjoint sets whose union has probability 1.

4. (Bernoulli process) Let $S = \mathbb{N}$, $0 < p < 1$,

$$\mathbb{P}(X_{n+1} = i+1|X_n = i) = p \quad \text{and} \quad \mathbb{P}(X_{n+1} = i|X_n = i) = 1-p,$$

   for every $n \geq 0$ and $i \in S$. The random variable X_n could count, for example, the number of heads in n tosses of a coin if we set $\mathbb{P}(X_0 = 0) = 1$. This is a homogeneous Markov chain with (infinite) transition matrix

$$T = \begin{bmatrix} 1-p & p & 0 & 0 & \cdots \\ 0 & 1-p & p & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

   i.e. for $i,j \in \mathbb{N}$

$$\pi_{i,j} = \begin{cases} 1-p, & i = j \\ p, & i+1 = j \\ 0, & \text{o.c.} \end{cases}.$$

   Show that

$$\mathbb{P}(X_n = j|X_0 = i) = C_{j-i}^n\, p^{j-i}(1-p)^{n-j+i}, \quad 0 \leq j-i \leq n.$$

[1] Given a vector $v \in \mathbb{R}^n$ we use the notation $v \geq 0$ to mean that each coordinate v_i of v is ≥ 0.

5. Considering the context of the beginning of Section 4 (recurrence) of Chapter 10 (Markov chains), show that

$$t_i = \sum_{n \geq 1} \mathcal{X}_{\{t_i \geq n\}}.$$

6. Prove that the mean value of the number of times the chain visits the state $i \in S$ is

$$\mathbb{E}\left(V_i | X_0 = i\right) = \frac{1}{1 - r_i}, \quad \text{if } \ 0 \leq r_i < 1,$$

where $r_i := P(X_n = i \text{ for some } n \geq 1 \ | X_0 = i)$.

7. Let $i \in S$. Prove that:

   (a) $\tau_i = +\infty$ iff $\lim_{n \to +\infty} \pi_{i,i}^{(n)} = 0$.

   (b) If $\tau_i = +\infty$, then for any $j \in S$, $\lim_{n \to +\infty} \pi_{j,i}^{(n)} = 0$.

8. Prove that:

   (a) $i \in R_+$ iff $\sum_n \pi_{i,i}^{(n)} = +\infty$ and $\lim_{n \to +\infty} \pi_{i,i}^{(n)} \neq 0$.

   (b) $i \in R_0$ iff $\sum_n \pi_{i,i}^{(n)} = +\infty$ and $\lim_{n \to +\infty} \pi_{i,i}^{(n)} = 0$.

   (c) $i \in T$ iff $\sum_n \pi_{i,i}^{(n)} < +\infty$ $\left(\text{in particular } \lim_{n \to +\infty} \pi_{i,i}^{(n)} = 0\right)$.

9. Consider $i \neq j$. Show that $i \to j$ is equivalent to $\displaystyle\sum_{n=1}^{+\infty} \mathbb{P}(t_j = n | X_0 = i) > 0$.

10. If X_n is an irreducible Markov chain with period d, is $Y_n = X_{nd}$ aperiodic?

11. Find a stationary distribution, and decide if it is unique, for the homogeneous Markov chain with state space $S = \{1, 2, 3, 4, 5, 6\}$ and transition probability matrix

$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

12. Find the unique stationary distribution for the homogeneous Markov chain with state space $S = \{1, 2, 3\}$ and transition probability matrix

$$T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix},$$

and compute the mean recurrence time of each state $i \in S$.

13. Consider a homogeneous Markov chain defined on the state space $S = \{1, 2\}$ with transition probability matrix

$$T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}.$$

Classify the states of the chain and determine their mean recurrence times:

(a) using the stationary distribution.

(b) computing the probability of first return after n steps.

14. Consider a homogeneous Markov chain on the state space $S = \mathbb{N}$ given by the transition probabilities:

$$\begin{aligned} \mathbb{P}(X_1 = i | X_0 = i) &= r, && i \geq 2, \\ \mathbb{P}(X_1 = i-1 | X_0 = i) &= 1 - r, && i \geq 2, \\ \mathbb{P}(X_1 = j | X_0 = 1) &= \frac{1}{2^j}, && j \geq 1. \end{aligned}$$

Classify the states of the chain and find their mean recurrence times:

(a) Using the stationary distribution.

(b) * Computing the probability of first return after n steps.

15. Determine the decomposition of the state space $S = \{1, 2, 3, 4, 5, 6\}$ of the homogeneous Markov chain with transition probability matrix

$$T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and compute the mean recurrence time, $\tau_i = \mathbb{E}(t_i | X_0 = i)$, of each state $i \in S$.

16. Consider a homogeneous Markov chain defined in the state space $S = \{1, 2, 3, 4, 5, 6\}$ with transition probability matrix

$$T = \begin{bmatrix} \frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \end{bmatrix}.$$

(a) Determine the decomposition of the state space of the chain and classify each state.

(b) Compute the period of each sate $i \in S$.

(c) Compute $\tau_6 = \mathbb{E}(t_6 | X_0 = 6)$.

17. Classify the states of the homogeneous Markov chains given by the transition matrices below, and determine the mean recurrence times of each state.

(a) $$\begin{bmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{bmatrix}, \text{ with } 0 < p < \tfrac{1}{2}.$$

(b) $$\begin{bmatrix} 0 & p & 0 & 1-p \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ p & 0 & 1-p & 0 \end{bmatrix}, \text{ with } 0 < p < 1.$$