Master in Mathematical Finance
Probability Theory and Stochastic Processes
First Semester - 2025/2026

List 8 - Martingales

1. Let $X_1, X_2, \ldots$ be a martingale with respect to the filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$. Show that:

   (a) If $X_0 = E(X_1)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$, then $X_0, X_1, X_2, \ldots$ is a martingale with respect to the filtration $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots$.

   (b) X_n is a martingale with respect to $\sigma(X_1, \ldots, X_n)$.

2. Let $Y_1, Y_2, \ldots$ be independent random variables such that

$$P(Y_n = a_n) = \frac{1}{2n^2}, \qquad P(Y_n = 0) = 1 - \frac{1}{n^2} \qquad \text{and} \qquad P(Y_n = -a_n) = \frac{1}{2n^2},$$

   where $a_1 = 2$, $a_n = 4 \sum_{j=1}^{n-1} a_j$. Decide if X_n and $\sigma(Y_1, \ldots, Y_n)$ define a martingale when:

   (a) $X_n = \sum_{j=1}^{n} Y_j$.

   (b) $X_n = \sum_{j=1}^{n} \frac{1}{2^j} Y_j$.

   (c) $X_n = \sum_{j=1}^{n} Y_j^2$.

3. Let $Y_1, Y_2, \ldots$ be a sequence of iid random variables such that $P(Y_n = 1) = p$ and $P(Y_n = -1) = 1 - p$. Let $S_n = \sum_{j=1}^{n} Y_j$. Decide if X_n and $\sigma(Y_1, \ldots, Y_n)$ define a martingale when

   (a) $X_n = S_n$.

   (b) $X_n = S_n^2 - n$.

   (c) $X_n = (-1)^n \cos(\pi S_n)$.

   (d) $X_n = \left(\frac{1-p}{p}\right)^{S_n}$.

   (e) $X_n = S_n - (2p - 1)n$.

4. Let $Y_1, Y_2, \ldots$ be a sequence of iid random variables with Poisson distribution and mean value λ. Consider also the sequence

$$X_n = X_{n-1} + Y_n - 1, \quad n \in \mathbb{N},$$

   and $X_0 = 0$. Find the values of λ for which X_n is a martingale, sub-martingale or super-martingale, with respect to the filtration $\sigma(Y_1, \ldots, Y_n)$.

5. Let X_n be a martingale with respect to a filtration $\mathcal{F}_n$. Prove that

$$E(X_{n+j} | \mathcal{F}_n) = X_n, \quad \text{for all } n, j \in \mathbb{N}.$$

6. Let X_n be a martingale with respect to the filtration $\mathcal{F}_n$ and τ is a stopping time. Determine $E(X_{\tau\wedge n})$.

7. Let $Y_1, Y_2, \dots$ be a sequence of iid random variables with distribution

$$P(Y_n = 1) = p \quad \text{and} \quad P(Y_n = -1) = 1 - p, \text{ where } 0 < p < 1 \text{ and } X_n = \sum_{j=1}^{n} Y_j.$$

Compute $E(\tau)$ for the stopping time

$$\tau = \min\{n \geq 1 \colon X_n = 1\}$$

when:

(a) $p \leq 1/2$.
Hint: Use Wald's equation (when $p < 1/2$ try also using the optional stopping theorem for $Z_n = [(1-p)/p]^{X_n}$).

(b) * $p > 1/2$.
Hint: Use the optional stopping theorem for $Z_n = X_n - (2p-1)n$. Look first at an application of the optional stopping theorem for $Z_{\tau\wedge n}$ in order to show that $E(\tau \wedge n)$ is bounded. Then take the same conclusion for $E(\tau)$.