Simplex Method – Complete Resolution of Example 2

Original Problem

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2, \\ \text{s. to} \quad & x_1 - 3x_2 \leq 3 \\ & -2x_1 + x_2 \leq 2 \\ & -3x_1 + 4x_2 \leq 12 \\ & 3x_1 + x_2 \leq 9 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Standard Form

We introduce slack variables x_3, x_4, x_5, x_6 to convert inequalities into equalities:

$$\begin{aligned} \max \quad & z = 5x_1 + 4x_2, \\ \text{s. to} \quad & x_1 - 3x_2 + x_3 = 3 \\ & -2x_1 + x_2 + x_4 = 2 \\ & -3x_1 + 4x_2 + x_5 = 12 \\ & 3x_1 + x_2 + x_6 = 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{aligned}$$

Initial Simplex Table (Iteration 0)

Basic Var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	1	-3	1	0	0	0	3
x_4	-2	1	0	1	0	0	2
x_5	-3	4	0	0	1	0	12
x_6	3	1	0	0	0	1	9
$\bar{z}$	-5	-4	0	0	0	0	0

The current solution is $x = (0, 0, 3, 2, 12, 9)$ with value $z = 0$.

Entering variable: x_1 (has the most negative coefficient in row $\bar{z}$), corresponds to column with $\min\{\bar{z}_j, \text{ for all } j \text{ such that variable } x_j \text{ is non basic}\} = \min\{-5, -4\} = 5$.

Leaving variable: Compute minimum ratio, for values strictly positive in column of x_1 (the entering variable):

$$\min\{\frac{\text{RHS from row } i}{\text{value in column of } x_1 \text{ in row } i} \text{ such that value in column of } x_1 \text{ and in row } i>0\} =$$

$$\min\{\frac{3}{1}, \quad \frac{9}{3}\} = 3 \text{ its a tie, choose one} \Rightarrow x_3 \text{ leaves the basis.}$$

Name the rows as follow:

	Basic Var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
row R_1	x_3	$\boxed{1}$	-3	1	0	0	0	3
row R_2	x_4	-2	1	0	1	0	0	2
row R_3	x_5	-3	4	0	0	1	0	12
row R_4	x_6	3	1	0	0	0	1	9
row R_5	$\bar{z}$	-5	-4	0	0	0	0	0

Iteration 1 – Pivot is in first row R_1 (row of x_3) and column 1 (column of x_1): x_1 enters, x_3 leaves

Pivot row: divide the row of the pivot by the pivot to make pivot = 1:

$$R^{'}_{\text{pivot}} := \frac{R_{\text{pivot}}}{\text{pivot}}$$

So, in this example, for R_1,

$$R^{'}_1 := \frac{R_1}{1}$$

All other rows: perform operations to make other elements in the column of the pivot = 0:

$$R^{'}_j := R_j - \frac{\text{element, in column of the pivot, to make 0}}{\text{pivot}} \times R_{\text{pivot}}$$

So, in this example,

$$R^{'}_2 := R_2 - \frac{-2}{1} \times R_1 = R_2 + 2 \times R_1$$

$$R^{'}_3 := R_3 - \frac{-3}{1} \times R_1 = R_3 + 3 \times R_1$$

$$R^{'}_4 := R_4 - \frac{3}{1} \times R_1 = R_4 - 3 \times R_1$$

$$R^{'}_5 := R_5 - \frac{-5}{1} \times R_1 = R_5 + 5 \times R_1$$

After pivotal operations:

x_B	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_1	1	-3	1	0	0	0	3
x_4	0	-5	2	1	0	0	8
x_5	0	-5	3	0	1	0	21
x_6	0	10	-3	0	0	1	0
$\bar{z}$	0	-19	5	0	0	0	15

The current solution is $x = (3, 0, 0, 8, 21, 0)$ with value $z = 15$.
Not all coefficients in row $\bar{z}$ are now non-negative. Therefore, the current solution is not yet optimal.
Entering variable: x_2 (most negative coefficient in row $\bar{z}$)
Leaving variable: Compute minimum ratio, for values strictly positive in column of x_2 (the entering variable):

$$\min\{\frac{0}{10}\} = 0 \Rightarrow x_6 \text{ leaves the basis}$$

Iteration 2 – Pivot is in row R_4 (row of x_6) and column 2 (column of x_2): x_2 enters, x_6 leaves

Pivot row: divide the row of the pivot by the pivot to make pivot = 1:

$$R^{'}_{\text{pivot}} := \frac{R_{\text{pivot}}}{\text{pivot}}$$

So, in this example, for R_4,

$$R^{'}_4 := \frac{R_4}{10}$$

All other rows: perform operations to make other elements in the column of the pivot = 0:

$$R_j' := R_j - \frac{\text{element to make 0}}{\text{pivot}} \times R_{\text{pivot}}$$

So, in this example,

$$R_1' := R_1 - \frac{-3}{10} \times R_4 = R_1 + \frac{3}{10} \times R_4$$

$$R_2' := R_2 - \frac{-5}{10} \times R_4 = R_2 + \frac{5}{10} \times R_4$$

$$R_3' := R_3 - \frac{-5}{10} \times R_4 = R_3 + \frac{5}{10} \times R_4$$

$$R_5' := R_5 - \frac{-19}{10} \times R_4 = R_5 + \frac{19}{10} \times R_4$$

After pivot operations:

x_B	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_1	1	0	$\frac{1}{10}$	0	0	$\frac{3}{10}$	3
x_4	0	0	$\frac{5}{10}$	1	0	$\frac{5}{10}$	8
x_5	0	0	$\frac{15}{10}$	0	1	$\frac{5}{10}$	21
x_2	0	1	$-\frac{3}{10}$	0	0	$\frac{1}{10}$	0
$\bar{z}$	0	0	$-\frac{7}{10}$	0	0	$\frac{19}{10}$	15

The current solution is $x = (3, 0, 0, 8, 21, 0)$ with value $z = 15$. Note that this is exactly the same solution that was obtained in the previous iteration. (This is because it is a degenerate solution. A degenerate solution can be identified by having a basic variable with value equal to zero.)

Not all coefficients in row $\bar{z}$ are now non-negative. Therefore, the current solution is not yet optimal.

Entering variable: x_3 (most negative coefficient in row $\bar{z}$)

Leaving variable: Compute minimum ratio, for values strictly positive in column of x_3 (the entering variable):

$$\min\{\frac{3}{\frac{1}{10}}, \quad \frac{8}{\frac{5}{10}}, \quad \frac{21}{\frac{15}{10}}\} = \min\{30, 16, 14\} = 14 \Rightarrow x_5 \text{ leaves the basis}$$

Iteration 3 – Pivot is in row 3 (row of x_5) and column 3 (column of x_3): x_3 enters, x_5 leaves

Pivot row: divide the row of the pivot by the pivot to make pivot = 1:

$$R_{\text{pivot}}' := \frac{R_{\text{pivot}}}{\text{pivot}}$$

So, in this example, for R_3,

$$R_3' := \frac{R_3}{\frac{3}{2}} = \frac{2}{3} \times R_3$$

All other rows: perform operations to make other elements in the column of the pivot = 0:

$$R_j' := R_j - \frac{\text{element to make 0}}{\text{pivot}} \times R_{\text{pivot}}$$

So, in this example,

$$R_1' := R_1 - \frac{\frac{1}{10}}{\frac{3}{2}} \times R_3 = R_1 - \frac{1}{15} R_3$$

$$R_2' := R_2 - \frac{\frac{5}{10}}{\frac{3}{2}} \times R_3 = R_2 - \frac{1}{3} \times R_3$$

$$R_4' := R_4 - \frac{-\frac{3}{10}}{\frac{3}{2}} \times R_3 = R_4 + \frac{1}{5} R_3$$

$$R_5' := R_5 - \frac{-\frac{7}{10}}{\frac{3}{2}} \times R_3 = R_5 + \frac{7}{15} \times R_3$$

After pivot operations:

x_B	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_1	1	0	0	0	$-\frac{1}{15}$	$\frac{4}{15}$	$\frac{8}{5}$
x_4	0	0	0	1	$-\frac{1}{3}$	$\frac{1}{3}$	1
x_3	0	0	1	0	$\frac{2}{3}$	$\frac{1}{2}$	14
x_2	0	1	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{21}{5}$
$\bar{z}$	0	0	0	0	$\frac{7}{15}$	$\frac{32}{15}$	$\frac{124}{5}$

The current solution is $x = (\frac{8}{5}, \frac{21}{5}, 14, 1, 0, 0)$ with value $z = \frac{124}{5}$.
All coefficients in row $\bar{z}$ are now non-negative. Therefore, the current solution is optimal.

If, in the first iteration, we select variable x_2 instead of x_1 to enter the basis, we would get the following:

Initial Simplex Table (Iteration 0)

Basic Var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	1	-3	1	0	0	0	3
x_4	-2	1	0	1	0	0	2
x_5	-3	4	0	0	1	0	12
x_6	3	1	0	0	0	1	9
$\bar{z}$	-5	-4	0	0	0	0	0

The current solution is $x = (0, 0, 3, 2, 12, 9)$ with value $z = 0$.
Entering variable: x_2 (most negative coefficient in row z)
Leaving variable: Compute minimum ratio, for values strictly positive in column of x_2 (the entering variable):

$$\min\{\frac{\text{RHS from row } j}{\text{value in column } x_2 \text{ in row } j}\} =$$

$$\min\{\frac{2}{1}, \quad \frac{12}{4}, \quad \frac{9}{1}\} = \min\{2, \quad 3, \quad 9\} = 2 \Rightarrow x_4 \text{ leaves the basis}$$

Iteration 1 – Pivot is in row 2 (row of x_4) and column 2 (column of x_2): x_2 enters, x_4 leaves

Name the rows as follow:

	Basic Var	x_1	x_2	x_3	x_4	x_5	x_6	RHS
row R_1	x_3	1	-3	1	0	0	0	3
row R_2	x_4	-2	$\boxed{1}$	0	1	0	0	2
row R_3	x_5	-3	4	0	0	1	0	12
row R_4	x_6	3	1	0	0	0	1	9
row R_5	$\bar{z}$	-5	-4	0	0	0	0	0

Pivot row: divide the row of the pivot by the pivot to make pivot = 1:
in this example, for R_2,

$$R_2' := \frac{R_2}{1}$$

All other rows: perform operations to make other elements in the column of the pivot = 0:

$$R_1' := R_1 - \frac{-3}{1} \times R_2 = R_1 + 3 \times R_2$$

$$R_3' := R_3 - \frac{4}{1} \times R_2 = R_3 - 4 \times R_2$$

$$R_4' := R_4 - \frac{1}{1} \times R_2 = R_4 - R_2$$

$$R_5' := R_5 - \frac{-4}{1} \times R_2 = R_5 + 4 \times R_2$$

After pivot operations:

x_B	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	-5	0	1	3	0	0	9
x_2	-2	1	0	1	0	0	2
x_5	5	0	0	-4	1	0	4
x_6	5	0	0	-1	0	1	7
$\bar{z}$	-13	0	0	4	0	0	8

The current solution is $x = (0, 2, 9, 0, 4, 7)$ with value $z = 8$.
Not all coefficients in row $\bar{z}$ are now non-negative. Therefore, the current solution is not yet optimal.
Entering variable: x_1 (most negative coefficient in row $\bar{z}$)
Leaving variable: Compute minimum ratio, for values strictly positive in column of x_1 (the entering variable):

$$\min\{\frac{\text{RHS from row } j}{\text{value in column } x_2 \text{ in row } j}\} =$$

$$\min\{\frac{4}{5}, \quad \frac{7}{5}\} = \frac{4}{5} \Rightarrow x_5 \text{ leaves the basis}$$

Iteration 2 – Pivot is in row 3 (row of x_5) and column 1 (column of x_1): x_1 enters, x_5 leaves

Pivot row: divide the row of the pivot by the pivot to make pivot = 1:
in this example, for R_3,

$$R_3^{'} := \frac{R_3}{5}$$

All other rows: perform operations to make other elements in the column of the pivot = 0:

$$R_1^{'} := R_1 - \frac{-5}{5} \times R_3 = R_1 + R_3$$

$$R_2^{'} := R_2 - \frac{-2}{5} \times R_3 = R_2 + \frac{2}{5} \times R_3$$

$$R_4^{'} := R_4 - \frac{5}{5} \times R_3 = R_4 - R_3$$

$$R_5^{'} := R_5 - \frac{-13}{5} \times R_3 = R_5 + \frac{13}{5} \times R_3$$

After pivot operations:

x_B	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	0	1	-1	1	0	13
x_2	0	1	0	$-\frac{3}{5}$	$\frac{2}{5}$	0	$\frac{18}{5}$
x_1	1	0	0	$-\frac{4}{5}$	$\frac{1}{5}$	0	$\frac{4}{5}$
x_6	0	0	0	3	-1	1	3
$\bar{z}$	0	0	0	$-\frac{32}{5}$	$\frac{13}{5}$	0	$\frac{84}{5}$

The current solution is $x = (\frac{4}{5}, \frac{18}{5}, 13, 0, 0, 3)$ with value $z = \frac{92}{5}$.
Not all coefficients in row $\bar{z}$ are now non-negative. Therefore, the current solution is not yet optimal.
Entering variable: x_4 (most negative coefficient in row $\bar{z}$)

Leaving variable: Compute minimum ratio, for values strictly positive in column of x_4 (the entering variable):

$$\min\{\frac{\text{RHS from row } j}{\text{value in column } x_4 \text{ in row } j}\} =$$

$$\min\{\frac{3}{3}\} = 1 \Rightarrow x_6 \text{ leaves the basis}$$

Iteration 3 – Pivot is in row 4 (row of x_6) and column 4 (column of x_4): x_4 enters, x_6 leaves

Pivot row: divide the row of the pivot by the pivot to make pivot = 1:
in this example, for R_4,

$$R_4^{'} := \frac{R_4}{3}$$

All other rows: perform operations to make other elements in the column of the pivot = 0:

$$R_1^{'} := R_1 - \frac{-1}{3} \times R_4 = R_1 + \frac{1}{3}R_4$$

$$R_2^{'} := R_2 - \frac{-\frac{3}{5}}{3} \times R_4 = R_2 + \frac{1}{5} \times R_4$$

$$R_3^{'} := R_3 - \frac{-\frac{4}{5}}{3} \times R_4 = R_3 + \frac{4}{15}R_4$$

$$R_5^{'} := R_5 - \frac{-\frac{32}{5}}{3} \times R_4 = R_5 + \frac{32}{15} \times R_4$$

After pivot operations:

x_B	x_1	x_2	x_3	x_4	x_5	x_6	RHS
x_3	0	0	1	0	$\frac{2}{3}$	$\frac{1}{2}$	14
x_2	0	1	0	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{21}{5}$
x_1	1	0	0	0	$-\frac{1}{15}$	$\frac{4}{15}$	$\frac{24}{15} = \frac{8}{5}$
x_4	0	0	0	1	$-\frac{1}{3}$	$\frac{1}{3}$	1
$\bar{z}$	0	0	0	0	$\frac{7}{15}$	$\frac{32}{15}$	$\frac{124}{5}$

The current solution is $x = (\frac{8}{5}, \frac{21}{5}, 14, 1, 0, 0)$ with value $z = \frac{124}{5}$.
All coefficients in row $\bar{z}$ are now non-negative. Therefore, the current solution is optimal.