

Lecture 4: Capital and Government

Cole, Chapters 14, 16 and 18

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Putting Capital in

- We will assume that the final good (output) can now be used either in the form of consumption C_t or in the form of investment X_t .
- We will denote the household's beginning-of-period capital stock as K_t so that the date t state has date t variables.
- The equation of motion for the capital stock is

$$K_{t+1} = (1 - \delta)K_t + X_t.$$

Putting Capital In

- Households produce the final good according to a standard Cobb-Douglas production function which is subject to a labor-augmenting productivity growth factor Z_t . With this, the resource constraint for the economy is given by

$$[Z_t L_t]^{1-\alpha} K_t^\alpha = C_t + X_t.$$

- Labor augmenting technical change $Z_t L_t$ which is equivalent to total factor technical change

$$Z_t^{1-\alpha} = \tilde{Z}_t$$

- Along BG path capital will grow at same rate as Z_t not $\tilde{Z}_t$. (Hence so will output.)

Putting Capital In

- The Cobb-Douglas production function $[Z_t L_t]^{1-\alpha} K_t^\alpha$ has many nice features.
 - First, if labor and capital double, then so does output; this is called constant returns to scale.
 - The marginal products are given by

$$\begin{aligned} MPL &= (1 - \alpha) Z_t L_t^{-\alpha} \frac{K_t^\alpha}{Z_t^\alpha}, \\ MPK &= \alpha \left[\frac{Z_t L_t}{K_t} \right]^{1-\alpha}. \end{aligned}$$

If labor is constant and both capital and productivity grow at the same rate, then the marginal product of capital, is constant while the marginal product of labor grows with productivity. This will turn out to lead to nice long-run growth predictions.

- The Cobb-Douglas production function $[Z_t L_t]^{1-\alpha} K_t^\alpha$ has many nice features.
 - Also, the shares paid out to labor and capital, which are

$$\begin{aligned}MPL * L &= (1 - \alpha) Y_t \\MPK * K &= \alpha * Y_t,\end{aligned}$$

are constant over time.

Putting Capital In

- The Cobb-Douglas production function $[Z_t L_t]^{1-\alpha} K_t^\alpha$ has many nice features.
 - Finally, when we look at how capital and labor inputs respond to price changes; since $MPL = w$, the equilibrium rental rate of labor, and $MPK = r$, the equilibrium rental rate of capital, in an optimal program, it follows that

$$\frac{MPL}{MPK} = \frac{w}{r} = \frac{(1-\alpha)K}{\alpha L} \quad \text{and hence}$$
$$d \log \left(\frac{w}{r} \right) = d \log \left(\frac{K}{L} \right).$$

In words, a 1 percent change in the ratio of input prices changes the relative inputs by 1 percent. This is called a unit elasticity.

Putting Capital In

- Because we have a representative agent economy again (since there is only a single good and all households end up being the same), we will not distinguish between a household's choice of labor, investment, and capital, and the aggregate choices.
- This is a serious abuse of notation and we need to be careful not to confuse ourselves.
- The transition equation for capital holds at the level of the household. However, the resource constraint holds at the level of the economy.
- A household simply takes prices as given and chooses its optimal plan.

Putting Capital In

- Assume perfect foresight so HH's problem is choosing sequence $\{C_t, L_t, M_{t+1}, B_{t+1}, K_{t+1}\}_{t=1}^2$ so as to maximize

$$\max \sum_{t=1,2} \beta^{t-1} [u(C_t) - v(L_t)] + \beta^2 V(M_3, B_3, K_3)$$

subject to

$$M_t \geq P_t C_t \text{ and}$$

$$\begin{aligned} & P_t [Z_t L_t]^{1-\alpha} K_t^\alpha + [M_t - P_t C_t] + B_t + T_t \\ \geq & M_{t+1} + q_t B_{t+1} + P_t [K_{t+1} - (1-\delta)K_t] \text{ for all } t \leq 2. \end{aligned}$$

only consumption shows up in the household's cash-in-advance constraint because new investment is transacted in the asset market or HH has other and more efficient means of making large purchases such as capital.

- We replaced the investment in the BC using the transition equation for capital. The HH picks the new capital stock: K_{t+1} . The $\{M_1, B_1, K_1\}$ are given.

Putting Capital In

- The household's f.o.c.'s now include a new one for capital.
- The consumption condition is unchanged

$$\beta^{t-1} u'(C_t) - P_t [\lambda_t + \mu_t] = 0.$$

- The f.o.c. for L_t has changed:

$$-\beta^{t-1} v'(L_t) + \mu_t P_t Z_t^{1-\alpha} (1-\alpha) L_t^{-\alpha} K_t^{\alpha} = 0.$$

similar except that the MPL now has L and K .

- The f.o.c.s for money and bonds are also unchanged and are given by

$$-\mu_t + \lambda_{t+1} + \mu_{t+1} = 0$$

and

$$-\mu_t q_t + \mu_{t+1} = 0.$$

Putting Capital In

- New condition coming for choice of capital stock at the end of period

$$-\mu_t P_t + \mu_{t+1} P_{t+1} \left[[Z_{t+1} L_{t+1}]^{1-\alpha} \alpha K_{t+1}^{\alpha-1} + 1 - \delta \right] = 0.$$

- These f.o.c.'s along with the c.i.a. constraint, the budget constraint, the resource constraint and the market clearing conditions for money and bonds give the constraints that our equilibrium must satisfy.
- Once again, the presence of money growth and productivity growth means that our economy is not going to be stationary.

Putting Capital In

- Assume that both productivity and the money supply grow at constant rates and normalize initial values to get

$$Z_t = (1 + g)^{t-1},$$

and

$$M_t = (1 + \tau)^{t-1}$$

Putting Capital In

Conjecture that capital grows at the same rate as productivity and that labor is constant, which implies that output will grow at rate g since

$$Y_t = [Z_t L]^{1-\alpha} K_t^\alpha$$

$$Y_{t-1} = [Z_{t-1} L]^{1-\alpha} K_{t-1}^\alpha$$

and hence

$$Y_t = (1 + g) Y_{t-1} = Z_t Y_1$$

given our normalization of $Z_1 = 1$.

We will also **conjecture** that the cash-in-advance constraint holds as an equality.

Putting Capital In

Since capital is assumed to grow at the same rate as productivity,

$$K_t = (1 + g)K_{t-1} = Z_t K_1.$$

So investment is

$$X_t = (1 + g)K_t - (1 - \delta)K_t = (g + \delta)Z_t K_1.$$

Taking account of this in the resource constraint, we get that

$$\begin{aligned} [Z_t L]^{1-\alpha} K_t^\alpha &= C_t + [g + \delta] K_t \\ \implies Z_t L^{1-\alpha} K_1^\alpha &= C_t + [g + \delta] Z_t K_1. \end{aligned}$$

So, consumption must also grow at rate g too.

Since everything except labor grows at rate g ,

$$L^{1-\alpha} K_1^\alpha = C_1 + [g + \delta] K_1.$$

Putting Capital In

Turn next to our cash-in-advance constraint which we are again assuming holds with equality. This yields

$$P_t = \frac{M_t}{C_t} = \frac{(1 + \tau)}{(1 + g)} P_{t-1}.$$

This is essentially the same result we derived previously when output was just produced with labor.

Putting Capital In

Assuming $u(c) = \log c$ and $v(l) = \frac{l^{1+\gamma}}{1+\gamma}$, our f.o.c. for consumption is

$$\begin{aligned}\beta^{t-1} \frac{1}{C_t} &= [\lambda_t + \mu_t] P_t = [\lambda_t + \mu_t] \frac{M_t}{C_t} \\ \Rightarrow \beta^{t-1} &= [\lambda_t + \mu_t] (1 + \tau)^{t-1} M_1.\end{aligned}$$

So once again, we make a change in variables, setting

$$\tilde{\lambda}_t = \lambda_t \frac{(1 + \tau)^{t-1}}{\beta^{t-1}},$$

$$\tilde{\mu}_t = \mu_t \frac{(1 + \tau)^{t-1}}{\beta^{t-1}}.$$

This gives us

$$1 = [\tilde{\lambda}_t + \tilde{\mu}_t] M_1 = [\tilde{\lambda}_t + \tilde{\mu}_t]$$

with the normalization $M_1 = 1$, indicating that their sum is invariant and suggesting that again they both are.

Putting Capital In

The f.o.c. for labor can be written as

$$\begin{aligned}\beta^{t-1}(L_t)^\gamma &= \mu_t P_t Z_t^{1-\alpha} (1-\alpha) L_t^{-\alpha} K_t^\alpha \\ &= \mu_t \frac{(1+\tau)^{t-1}}{(1+g)^{t-1}} P_1 Z_1^{1-\alpha} (1-\alpha) L_t^{-\alpha} K_t^\alpha\end{aligned}$$

once we substitute in for P_t . This simplifies to

$$\beta^{t-1}(L)^{\gamma+\alpha} = \mu_t \frac{(1+\tau)^{t-1}}{(1+g)^{t-1}} P_1 Z_1^{1-\alpha} (1-\alpha) K_1^\alpha (1+g)^{t-1}$$

which, since $Z_1 = 1$, simplifies to

$$L^{\gamma+\alpha} = \tilde{\mu}_t P_1 (1-\alpha) K_1^\alpha$$

Putting Capital In

The f.o.c.'s for money and bonds can be written as

$$\tilde{\mu}_t = \frac{\beta}{1+\tau} [\tilde{\lambda}_{t+1} + \tilde{\mu}_{t+1}] = \frac{\beta}{1+\tau}$$

and

$$\tilde{\mu}_t q_t = \frac{\beta}{1+\tau} \tilde{\mu}_{t+1} \implies q_t = \frac{\beta}{1+\tau}.$$

These are exactly our standard results.

Putting Capital In

The f.o.c. for capital,

$$\mu_t P_t = \mu_{t+1} P_{t+1} \left([Z_{t+1} L_{t+1}]^{1-\alpha} \alpha K_{t+1}^{\alpha-1} + 1 - \delta \right),$$

can be rewritten as follows once we substitute out for prices

$$\mu_t = \mu_{t+1} \frac{1+\tau}{1+g} \left\{ \alpha \left[\frac{Z_{t+1} L_{t+1}}{K_{t+1}} \right]^{1-\alpha} + 1 - \delta \right\}.$$

since Z_t and K_t grow at the same rate and labor is constant

$$\mu_t = \mu_{t+1} \frac{1+\tau}{1+g} \left\{ \alpha \left[\frac{Z_1 L}{K_1} \right]^{1-\alpha} + 1 - \delta \right\}.$$

Finally we put in our change in variables to get that and $\tilde{\mu}_t = \tilde{\mu}_{t+1}$

$$1 = \beta \frac{1}{1+g} \left\{ \alpha \left[\frac{Z_1 L}{K_1} \right]^{1-\alpha} + 1 - \delta \right\}.$$

Putting Capital In

Our equations are: the RC

$$[Z_1 L]^{1-\alpha} K_1^\alpha = C_1 + [g + \delta] K_1.$$

The second is our optimal labor condition,

$$L^{\gamma+\alpha} = \frac{\beta}{1+\tau} P_1 (1-\alpha) Z_1^{1-\alpha} K_1^\alpha. \quad (1)$$

The third is our optimal capital condition,

$$1 = \frac{\beta}{1+g} \left\{ \alpha \left[\frac{Z_1 L}{K_1} \right]^{1-\alpha} + 1 - \delta \right\}. \quad (2)$$

The fourth is our optimal consumption condition,

$$1 = [\tilde{\lambda} + \tilde{\mu}].$$

Putting Capital In

The fifth is our optimal money condition,

$$\frac{\beta}{1+\tau} [\tilde{\lambda} + \tilde{\mu}] = \frac{\beta}{1+\tau} = \tilde{\mu}.$$

The sixth is our optimal bond condition, which pins down the discount rate,

$$q = \frac{\beta}{1+\tau}.$$

The last is our price condition from the c.i.a. constraint

$$P_1 = \frac{M_1}{C_1} = \frac{1}{C_1}$$

Putting Capital In

- We have seven variables to solve for $[C_1, L, P_1, \tilde{\mu}, \tilde{\lambda}, q, K_1]$ and we do so taking as given Z_1 and M_1 along with g and τ .
- The one surprise is that we have to "solve" for K_1 . This is because even the initial level of the capital stock must be consistent with balanced growth if we are going to be on a balanced growth path from the start, which we are assuming.
- Fortunately, our system is again block recursive so we can solve things in steps.

Putting Capital In

Step 1: Start with the resource constraint and note that it implies that

$$C_1 = [Z_1 L]^{1-\alpha} K_1^\alpha - [g + \delta] K_1,$$

hence if we know K_1 and L we know C_1 . Then, move to the price condition and note that we can use our expression for C_1 to get that

$$P_1 = \frac{M_1}{[Z_1 L]^{1-\alpha} K_1^\alpha - [g + \delta] K_1}.$$

Hence the initial price level is also determined by K_1 and L .

Putting Capital In

Step 2: We can solve for K_1 and L as a simultaneous block given these results. Start with our optimality condition for capital,

$$1 = \frac{\beta}{1+g} \left\{ \alpha \left[\frac{Z_1 L}{K_1} \right]^{1-\alpha} + 1 - \delta \right\}.$$

This depends only on K_1 and L to begin with. Moreover, it really just serves to pin down the ratio of productive labor - $Z_1 L$ - and capital K .

$$\frac{Z_1 L}{K_1} = \left\{ \frac{1}{\alpha} \left(\frac{1+g}{\beta} - 1 + \delta \right) \right\}^{1/(1-\alpha)}. \quad (3)$$

- Productive-labor-to-capital ratio depends on the discount rate, the level of productivity growth, and the depreciation rate, but not the rate of inflation.
- Surprisingly the capital-to-labor ratio is negatively affected by growth.

Putting Capital In

To simplify things further, substitute for P_1 and for $\tilde{\mu}$ in our optimal labor condition to get that

$$L^{\gamma+\alpha} = \left[\frac{\beta}{1+\tau} \frac{1}{M_1} \right] (1-\alpha) Z_1^{1-\alpha} K_1^\alpha \left[\frac{M_1}{[Z_1 L]^{1-\alpha} K_1^\alpha - [g + \delta] K_1} \right],$$

We already have pinned down the capital-to-labor ratio using (3). So note

$$\left[\frac{M_1}{[Z_1 L]^{1-\alpha} K_1^\alpha - [g + \delta] K_1} \right] \times \frac{L^{-1}}{L^{-1}} = \frac{M_1 L^{-1}}{[Z_1]^{1-\alpha} \left[\frac{K_1}{L} \right]^\alpha - [g + \delta] \frac{K_1}{L}}.$$

Plugging into the original equation, converting our last K_1 into K_1/L and simplifying yields

$$L^{1+\gamma} = \left[\frac{\beta}{1+\tau} \right] (1-\alpha) Z_1^{1-\alpha} \left[\frac{K_1}{L} \right]^\alpha \left[\frac{1}{[Z_1]^{1-\alpha} \left[\frac{K_1}{L} \right]^\alpha - [g + \delta] \frac{K_1}{L}} \right]. \quad (4)$$

Once we have K_1/L , just substitute here to get L .

Putting Capital In

$$L^{1+\gamma} = \left[\frac{\beta}{1+\tau} \right] (1-\alpha) Z_1^{1-\alpha} \left[\frac{K_1}{L} \right]^\alpha \left[\frac{1}{[Z_1]^{1-\alpha} \left[\frac{K_1}{L} \right]^\alpha - [g + \delta] \frac{K_1}{L}} \right].$$

Once again we see money growth depressing labor in essentially the exact same way as in our simple model, in which output was produced with labor.

Step 4: Once we have solved for L and K_1 , we can easily determine the rest of our variables.

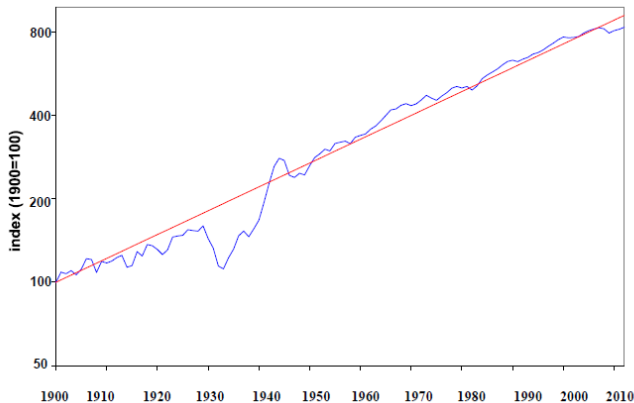
Putting Capital In

Kaldor's long Run growth facts were:

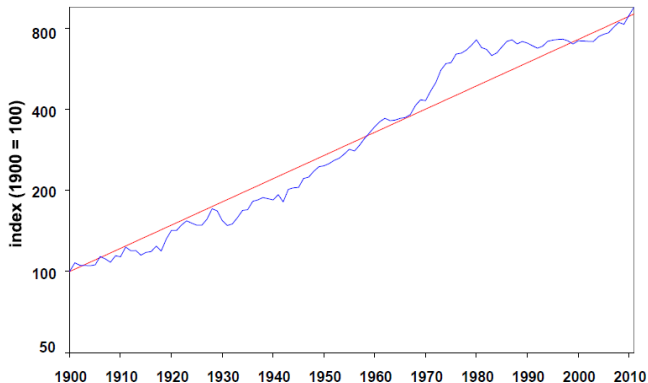
- 1 The growth rate of output per worker is roughly constant over time.
- 2 The growth rate of capital per worker is roughly constant over time but varies a lot across countries.
- 3 The capital-to-output ratio is roughly constant over time.
- 4 The return on investment is roughly constant over time.
- 5 The real wage grows over time.
- 6 The shares of national income paid to labor and capital are roughly constant over time.

Putting Capital In

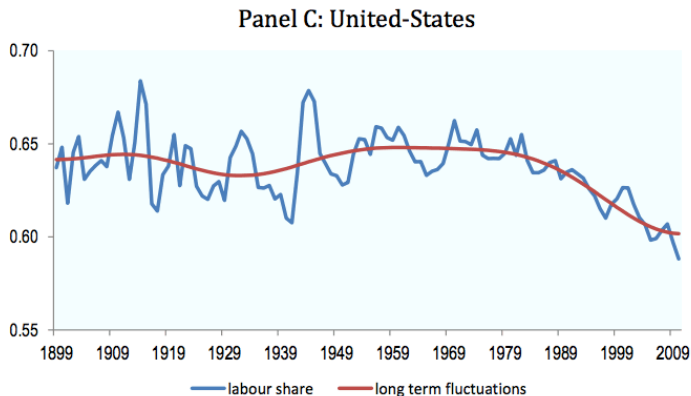
Real GDP per working-age person in the United States



Real GDP per working-age person in Brazil



Putting Capital In



Putting Taxes in

- Cannot study redistribution in a representative agent model.
- So just focus on the aggregate impact.
- Want to include both taxes on capital and labor.
- Cannot do this in model with only HH production - see through to overall tax consequences.
- Need to have each household employ outside labor and rent outside capital.
- Assume labor and capital rented in competitive markets with prices w and r .
- No government spending, lump-sum transfers give taxes back, but receive my share of total, *not my contribution to total*.

Putting Taxes in

- Let τ_l and τ_k denote the tax rates on capital and labor.
- Then the after-tax level of labor income is given by $(1 - \tau_l)w_tL_t$ and tax receipts are given by $\tau_lw_tL_t$.
- Capital taxation is normally done net of depreciation, but we are going to ignore that nicety.
- So, if r_t is the rental rate on capital, then after-tax capital income is given by $(1 - \tau_k)r_tK_t$ and tax receipts are given by $\tau_kr_tK_t$.
- The government's per capita transfers its receipts back to the public

$$T_t = \tau_lw_tL_t + \tau_kr_tK_t + \bar{M}_{t+1} - \bar{M}_t$$

- Taxes are collected in the Asset Market at the end of the period, and then transferred back.
- Since lump-sum rebated resource constraint unchanged.

Putting Taxes in

- A household running a production plant using hired labor $\bar{L}_t$ and capital $\bar{K}_t$ will just seek to maximize its net revenue which is given by

$$\Pi_t = P_t [Z_t \bar{L}_t]^{1-\alpha} \bar{K}_t^\alpha - w_t \bar{L}_t - r_t \bar{K}_t$$

- The household will optimally choose to set the marginal product equal to the rental price; i.e. $\partial \Pi_t / \partial \bar{L}_t = 0$ and $\partial \Pi_t / \partial \bar{K}_t = 0$
- If we assume that markets are competitive and hence that factors are paid their marginal products, we get that

$$w_t = P_t \left[(1 - \alpha) Z_t^{1-\alpha} \bar{L}_t^{-\alpha} \bar{K}_t^\alpha \right]$$

and that

$$r_t = P_t \left[\alpha [Z_t \bar{L}_t]^{1-\alpha} \bar{K}_t^{\alpha-1} \right]$$

- Putting this together with our taxes yields a different income term in the household's budget constraint in the asset market.

Putting Taxes in

- But wait, note that

$$P_t \left[(1 - \alpha) Z_t^{1-\alpha} L_t^{-\alpha} K_t^\alpha \right] L_t = (1 - \alpha) P_t Y_t$$

and that

$$P_t \left[\alpha [Z_t L_t]^{1-\alpha} K_t^{\alpha-1} \right] K_t = \alpha P_t Y_t$$

Putting this together would imply that the output term Y_t in the old budget constraint is now replaced by

$$[(1 - \alpha)(1 - \tau_l) + \alpha(1 - \tau_k)] [Z_t L_t]^{1-\alpha} K_t^\alpha.$$

- But then, what is the difference between labor and capital taxes?
- The answer is that in making its labor decision a household takes w_t and r_t as given and does not internalize the impact its choices would have on these returns.
- If all households were integrated and hence owned their own capital and worked in the family business, there would not be much difference between the two taxes. They would all just boil down to a tax on output.

Putting Taxes in

- In light of this remark, we are going to need to distinguish very carefully between the aggregate levels of capital and labor, $\bar{K}_t$ and $\bar{L}_t$, over which the household has no control, and the household's choice variables over which it does have control.
- The marginal products will depend only on the aggregates. Of course in equilibrium $\bar{K}_t = K_t$ and $\bar{L}_t = L_t$.
- So rental rates are

$$w_t = P_t \left[(1 - \alpha) Z_t^{1-\alpha} \bar{L}_t^{-\alpha} \bar{K}_t^{\alpha} \right]$$

and

$$r_t = P_t \left[\alpha [Z_t \bar{L}_t]^{1-\alpha} \bar{K}_t^{\alpha-1} \right]$$

Putting Taxes in

Given this, we can write the household's choice problem as choosing a sequence of quantities $\{C_t, L_t, M_{t+1}, B_{t+1}, K_{t+1}\}_{t=1}^2$ so as to maximize

$$\max \sum_{t=1,2} \beta^{t-1} [u(C_t) - v(L_t)] + \beta^2 V(M_3, B_3, K_3)$$

subject to

$$M_t \geq P_t C_t \text{ and}$$

$$\begin{aligned} & \Pi_t + P_t \{ (1 - \tau_l) w_t L_t + (1 - \tau_k) r_t K_t - \delta K_t \} \\ & + [M_t - P_t C_t] + B_t + T_t \\ \geq & M_{t+1} + q_t B_{t+1} + P_t [K_{t+1} - K_t] \text{ for all } t \leq 2. \end{aligned}$$

- With CRTS then $\Pi_t = 0$.
- Only labor and capital affected by our new rental rates.
- Only labor and capital foc's affected by the tax policy variables.

Putting Taxes in

In the book the HH problem is written like this, but you have to carefully distinguish between aggregate and household variables.

$$\max \sum_{t=1,2} \beta^{t-1} [u(C_t) - v(L_t)] + \beta^2 V(M_3, B_3, K_3)$$

subject to

$$M_t \geq P_t C_t \text{ and}$$

$$\begin{aligned} & P_t \left\{ \begin{array}{l} (1 - \tau_l) \left[(1 - \alpha) Z_t^{1-\alpha} \bar{L}_t^{-\alpha} \bar{K}_t^\alpha \right] L_t + \\ (1 - \tau_k) \left[\alpha [Z_t \bar{L}_t]^{1-\alpha} \bar{K}_t^{\alpha-1} \right] K_t - \delta K_t \end{array} \right\} \\ & + [M_t - P_t C_t] + B_{t-1} + T_t \\ & \geq M_{t+1} + q_t B_{t+1} + P_t [K_{t+1} - K_t] \text{ for all } t \leq 2. \end{aligned}$$

Putting Taxes in

The f.o.c. for labor is given by

$$-\beta^{t-1}v'(L_t) + \mu_t P_t (1 - \tau_l) Z_t^{1-\alpha} (1 - \alpha) \bar{L}_t^{-\alpha} \bar{K}_t^{\alpha} = 0.$$

The f.o.c. for capital is given by

$$-\mu_t P_t + \mu_{t+1} P_{t+1} \left[(1 - \tau_k) [Z_{t+1} \bar{L}_{t+1}]^{1-\alpha} \alpha \bar{K}_{t+1}^{\alpha-1} + 1 - \delta \right] = 0.$$

Here we see the tax wedges affecting the individual's choices. Moreover, each tax affects only its own marginal condition so labor and capital taxes here are playing a distinct role.

Putting Taxes in

If we undertake the same sort of analytics as we did in the balanced growth section, and set the aggregate equal to the individual choice values for capital and labor, we will get the following variants on our two conditions.

$$L^{\gamma+\alpha} = \tilde{\mu} P_1 (1 - \tau_l) (1 - \alpha) Z_1^{1-\alpha} K_1^\alpha \quad (5)$$

and

$$1 = \frac{\beta}{1+g} \left\{ (1 - \tau_k) \alpha \left[\frac{Z_1 L}{K_1} \right]^{1-\alpha} + 1 - \delta \right\}. \quad (6)$$

Remark: Equation (6) indicates that any change in the tax rate is offset by changes in the capital-to-labor ratio, and that the after-tax return on capital is pinned down by things such as the discount rate β and the growth rate g .

Putting Taxes in

We can use expression (6) to pin down the ratio of productive labor to capital just as we did without taxes,

$$\frac{Z_1 L}{K_1} = \left\{ \frac{1}{\alpha(1 - \tau_k)} \left(\frac{1 + g}{\beta} - 1 + \delta \right) \right\}^{1/(1-\alpha)}. \quad (7)$$

$$P_1 = \frac{M_1}{C_1}$$

and that

$$\tilde{\mu} = \frac{\beta}{1 + \tau} \frac{1}{M_1}.$$

Remark: $\tau_k \uparrow \Rightarrow \frac{L}{K_1} \uparrow \Rightarrow \frac{w_t}{P_t} \downarrow$

Putting Taxes in

We then use these two results just as we did before when we did not have taxes in our optimal labor condition to get that

$$L^\gamma = \left[\frac{\beta}{1 + \tau} \right] \left[\frac{1}{[Z_1 L]^{1-\alpha} K_1^\alpha - [g + \delta] K_1} \right] (1 - \tau_l)(1 - \alpha) Z_1^{1-\alpha} \left(\frac{K_1}{L} \right)^\alpha.$$

Now all we need to do is follow the same steps as in the case without taxes to get the analog of our labor condition

$$L^{1+\gamma} = \left[\frac{\beta}{1 + \tau} \right] \left[\frac{1}{[Z_1]^{1-\alpha} \left(\frac{K_1}{L} \right)^\alpha - [g + \delta] \frac{K_1}{L}} \right] (1 - \tau_l)(1 - \alpha) Z_1^{1-\alpha} \left(\frac{K_1}{L} \right)^\alpha. \quad (8)$$

and then we can just substitute in for the capital-to-labor ratio

Putting Taxes in

Some computational steps to our solution:

1. Compute K/L ratio via:

$$\frac{Z_1 L}{K_1} = \left\{ \frac{1}{\alpha(1-\tau_k)} \left(\frac{1+g}{\beta} - 1 + \delta \right) \right\}^{1/(1-\alpha)}.$$

2. Compute C_1/L ratio via

$$\frac{C_1}{L} = [Z_1]^{1-\alpha} \left(\frac{K_1}{L} \right)^\alpha - [g + \delta] \frac{K_1}{L}$$

3. Compute L via

$$L^{1+\gamma} = \left[\frac{\beta}{1+\tau} \right] \left[\frac{1}{C_1/L} \right] (1-\tau_l)(1-\alpha) Z_1^{1-\alpha} \left(\frac{K_1}{L} \right)^\alpha.$$

A higher K_1/L raises L directly through the MPL and the substitution effect. But also lowers it through C_1/L and the income effect.

Putting Taxes in

The capital-to-labor ratio is also important for labor productivity. To see this, note that

$$\frac{Y_t}{L_t} = \frac{Y_1(1+g)^{t-1}}{L}$$

where Y_1 is initial output when $t = 1$ and hence the level of productivity is Z . This then leads to

$$\frac{Y_1}{L} = \frac{(Z_1 L)^{1-\alpha} K_1^\alpha}{L} = Z_1^{1-\alpha} \left(\frac{K_1}{L} \right)^\alpha. \quad (9)$$

- From (7) we can see that capital taxes are important for labor productivity, but labor taxes are not.
- From (7) and (8) we can see that both capital taxes (through the capital-to-labor ratio) and labor taxes are important for the level of labor.

Problem

Use the following parameter settings:

$\gamma = 0.5$ Frisch elasticity of 2

$g = .01$ growth rate of labor productivity

$\beta = 1/1.02$ discount rate given risk-free rate

$\alpha = 1/3$ capital's share

$\tau = g$ growth rate of money, inflation is $1 + \tau/1 + g$

$\delta = .08$ depreciation rate of capital

- 1 Compute what happens if we have labor and capital taxes of 15 percent.
- 2 Compute what happens if we only raise this revenue through labor or through capital taxes.

Positive and Negative Views on Redistribution

Right now, governments are engaged in massive redistribution.

- We cannot analyze the positive aspects of redistribution
- But we can think about the waste aspect. Assume

$$T_t = \tau_l w_t L_t + \tau_k r_t K_t + \bar{M}_{t+1} - \bar{M}_t - WZ_t$$

where WZ_t is per capita waste.

- Then the resource constraint becomes

$$[Z_t L_t]^{1-\alpha} K_t^\alpha = C_t + X_t + Z_t W$$

- When we divide by Z_t and do some substitution, we now get

$$C_1 = L^{1-\alpha} K_1^\alpha - [g + \delta] K_1 - W$$

Extra Problem

Use the same parameter settings as the baseline in the Problem.

Compute the outcomes when waste is 0 and 10 percent of total taxes without waste.

Compare what happens.

Is the K/L ratio different? Is L or C or Y different?

Matlab and Mesh

Use the mesh plot function to create 3-D plots like in the text.

Construct your tax grid: $Tax = [0 : 0.1 : 0.8]$.

The loop through with i and j setting the capital tax $\tau_k = Tax(i)$ and the labor tax $\tau_l(j) = Tax(j)$.

For each of your tax values solve for various things like: $Labor(i, j)$ which gives labor given the capital tax and the labor tax.

Then plot things using $mesh(Tax, Tax, Labor)$. See matlab's documentation for more info.

You might want a finer tax grid when you doing the part with 0 taxes on capital or labor.

Putting Government in

Government spending can have many different positive benefits, for example:

- as a direct utility benefit, such as public television or national defense
- as a productive input such as roads and bridges and the legal system
- insurance through various transfers
- spending on education
- spending on basic research, which affects productivity growth

All of these are potentially important, but the easiest to accommodate are the first two channels. So, we will limit the roles we consider, while recognizing that, clearly, this is not anywhere close to the last word on this subject.

Putting Government in

- We will simplify our tax system so that taxes are one-dimensional and are simply a flat rate output tax τ_Y .
- We will also do away with government debt and assume that spending is just equal to revenue.
- This leads to the following form of the government budget constraint

$$\omega G_{t+1} = \tau_Y Y_t,$$

where $\omega > 1$ is our model of government waste or inefficiency, and revenue that is collected today funds expenditure in the next period.

- We will assume that government spending is divided into a fraction ψ of spending that works as a consumption good

and a fraction $1 - \psi$ that works as a production input.

Putting Government in

- This leads us to modify our flow utility to be

$$u(C) - v(L) + h(\psi G),$$

- and our production function to be

$$Y = f(ZL, K, (1 - \psi)G)$$

- To do anything quantitative, we are going to have to take a stand on $h(\cdot)$ and $f(\cdot)$.
- Clearly, we can get almost any kind of answer if we make "creative" enough assumptions. So thinking of a way to rationalize our assumptions and thereby discipline them vis-a-vis the data is going to be important.

Putting Government in

- With respect to preferences, we have gone with a constant-elasticity or power-function type of preferences. These preferences have nice properties.
- In addition, unless we want either consumption or government spending to crowd out the other, the functional form should be the same.
- This suggests that $u(\cdot) = a \times h(\cdot)$. This is going to lead us to log preferences over government consumption.

Putting Government in

- It is common to use constant-returns-to-scale and CES production functions, Cobb-Douglas, works well for private inputs.
- This suggests something like the following:

$$Y_t = \left\{ \omega \left[(Z_t L_t)^{1-\alpha} K_t^\alpha \right]^\rho + (1-\omega) [(1-\psi) G_t]^\rho \right\}^{1/\rho}. \quad (10)$$

- Note that if the cost of a unit of private value added is W_P and the cost of a unit of government input is W_G , then we get that

$$\begin{aligned} Y_t^{1-\rho} \omega P^{\rho-1} &= W_P, \\ Y_t^{1-\rho} (1-\omega) (1-\psi)^\rho G^{\rho-1} &= W_G. \end{aligned}$$

This implies that

$$d \log \left(\frac{W_P}{W_G} \right) = (\rho - 1) d \log \left(\frac{P}{G} \right).$$

Putting Government in

- A special case is $\rho = 0$, which corresponds to Cobb-Douglas and gives us the following production function:

$$Y = (ZL)^{1-\alpha-\nu} K^\alpha ((1-\psi)G)^\nu. \quad (11)$$

- If we assume log preferences over government consumption and a Cobb-Douglas aggregator over private and public inputs, we get a fairly tractable model with which to begin our analysis. This representation gives us a four parameter system within which to think about government spending.
- The household's problem is fairly standard since G is exogenous to the household.
- Since there is just an output tax, we can back to our basic model with HH production and no rental market for labor and capital.

Putting Government in

HH problem as choosing a sequence of quantities $\{C_t, L_t, M_t, B_{t+1}, K_{t+1}\}_{t=1}^2$ so as to maximize

$$\max \sum_{t=1,2} \beta^{t-1} [u(C_t) - v(L_t) + h(\psi G_t)] + \beta^2 V(M_3, B_3, K_3)$$

subject to

$$M_t \geq P_t C_t \text{ and}$$

$$\begin{aligned} & P_t \left[(1 - \tau_Y) [Z_t L_t]^{1-\alpha-\nu} K_t^\alpha ((1 - \psi) G_t)^\nu - \delta K_t \right] + [M_t - P_t C_t] + B_t + T_t \\ & \geq M_{t+1} + q_t B_{t+1} + P_t [K_{t+1} - K_t] \text{ for all } t \leq 2. \end{aligned}$$

Our first-order conditions are largely unchanged, but we do need to take account of the tax on output.

Putting Government in

The consumption condition is given by

$$\beta^{t-1} u'(C_t) - P_t [\lambda_t + \mu_t] = 0.$$

The f.o.c. for L_t is

$$-\beta^{t-1} v'(L_t) + \mu_t P_t (1 - \tau_Y) Z_t^{1-\alpha-\nu} (1 - \alpha - \nu) L_t^{-\alpha-\nu} K_t^\alpha ((1 - \psi) G_t)^\nu = 0.$$

The f.o.c. for capital is given by

$$0 = -\mu_t P_t + \mu_{t+1} P_{t+1} \times \\ \left[(1 - \tau_Y) [Z_{t+1} L_{t+1}]^{1-\alpha-\nu} ((1 - \psi) G_t)^\nu \alpha K_{t+1}^{\alpha-1} + 1 - \delta \right].$$

The f.o.c. for money is given by

$$-\mu_t + \lambda_{t+1} + \mu_{t+1} = 0$$

and the f.o.c for bonds is given by

Putting Government in

- These f.o.c.'s along with the c.i.a. constraint, the budget constraint, the resource constraint, and the market clearing conditions for money and bonds, give the constraints that our equilibrium must satisfy.
- Once again, the presence of money growth and productivity growth means that our economy is not going to be stationary.
- Money growth means that nominal balances will be nonstationary, productivity growth means that output and capital will be nonstationary, and the combination of these two effects means that prices will generally be nonstationary.
- As usual this will lead us to solve out for a balanced growth path.

Putting Government in

- To do this, we again assume that both productivity and the money supply grow at constant rates given by

$$Z_t = (1 + g)Z_{t-1}, \quad \text{with } Z_1 = 1,$$

and

$$M_t = (1 + \tau)M_{t-1} \quad \text{with } M_1 = 1.$$

- Conjecture that capital and government spending grow at the same rate as productivity and that labor is constant which implies that output will grow at rate g .
- This means that tax revenue grows at this rate and hence so does government spending, so these conjectures are at least consistent.

Putting Government in

- Since capital is assumed to grow at the same rate as productivity,

$$K_t = (1 + g)K_{t-1} = Z_t K_1.$$

- Making use of the fact that capital grows at rate g , we can pin down investment as

$$X_t = (1 + g)K_t - (1 - \delta)K_t = (g + \delta)Z_t K_1.$$

- Similarly, since

$$G_t = (1 + g)G_{t-1} = Z_t G_1.$$

Putting Government in

Taking account of this in the resource constraint, we start with

$$\begin{aligned}(1 - \tau_Y) [Z_t L]^{1-\alpha-\nu} K_t^\alpha ((1 - \psi) G_t)^\nu &= C_t + [(g + \delta)] K_t \\ \tau_Y [Z_t L]^{1-\alpha-\nu} K_t^\alpha ((1 - \psi) G_t)^\nu &= \omega G_{t+1} = \omega(1 + g) G_t \\ \implies Z_t \left\{ L^{1-\alpha-\nu} K_1^\alpha ((1 - \psi) G_1)^\nu &= C_1 + [g + \delta] Z_t K_1 + \omega(1 + g) G_1 \right\}\end{aligned}$$

Next, our cash-in-advance constraint (which we are assuming holds with equality) yields as before,

$$P_t = \frac{M_t}{C_t} = \frac{(1 + \tau)}{(1 + g)} P_{t-1}.$$

Putting Government in

- We can rewrite our f.o.c. for consumption as

$$\beta^{t-1} \frac{1}{C_t} = [\lambda_t + \mu_t] P_t = [\lambda_t + \mu_t] \frac{M_t}{C_t}.$$

- This leads to

$$\beta^{t-1} = [\lambda_t + \mu_t] (1 + \tau)^{t-1} M_1.$$

- So once again, we make a change in variables, setting

$$\tilde{\lambda}_t = \lambda_t \frac{(1 + \tau)^{t-1}}{\beta^{t-1}},$$

$$\tilde{\mu}_t = \mu_t \frac{(1 + \tau)^{t-1}}{\beta^{t-1}}.$$

- This gives us

$$1 = [\tilde{\lambda}_t + \tilde{\mu}_t],$$

indicating that their sum is invariant and suggesting that they both are.

Putting Government in

The f.o.c. for labor can be written as

$$\begin{aligned}\beta^{t-1} L_t^\gamma &= \mu_t (1 - \tau_Y) P_t Z_t^{1-\alpha-\nu} (1 - \alpha - \nu) L_t^{-\alpha-\nu} K_t^\alpha ((1 - \psi) G_t)^\nu \\ \implies L_t^\gamma &= \tilde{\mu}_t (1 - \tau_Y) \frac{1}{C_1} (1 - \alpha - \nu) L_t^{-\alpha-\nu} K_1^\alpha ((1 - \psi) G_1)^\nu \\ \implies L^{\gamma+\alpha+\nu} &= \frac{\beta}{1 + \tau} \frac{(1 - \tau_Y)}{C_1} (1 - \alpha - \nu) K_1^\alpha ((1 - \psi) G_1)^\nu,\end{aligned}$$

where I have dropped the time subscript for labor, since clearly it will be constant on our balanced growth path.

Putting Government in

The f.o.c. for capital,

$$\mu_t P_t = \mu_{t+1} P_{t+1} \left((1 - \tau_Y) [Z_{t+1} L_{t+1}]^{1-\alpha-\nu} \alpha K_{t+1}^{\alpha-1} ((1 - \psi) G_{t+1})^\nu + 1 - \delta \right),$$

can be rewritten as follows once we substitute out for prices

$$\mu_t = \mu_{t+1} \frac{1 + \tau}{1 + g} \left\{ (1 - \tau_Y) \alpha [Z_{t+1} L_{t+1}]^{1-\alpha-\nu} K_{t+1}^{\alpha-1} ((1 - \psi) G_t)^\nu + 1 - \delta \right\}.$$

since Z_t , G_t and K_t all grow at the same rate and L is constant

$$\mu_t = \mu_{t+1} \frac{1 + \tau}{1 + g} \left\{ (1 - \tau_Y) \alpha L^{1-\alpha-\nu} K_1^{\alpha-1} ((1 - \psi) G_1)^\nu + 1 - \delta \right\}.$$

Finally, we put in our change of variables to get that

$$1 = \frac{\beta}{1 + g} \left\{ (1 - \tau_Y) \alpha L^{1-\alpha-\nu} K_1^{\alpha-1} ((1 - \psi) G_1)^\nu + 1 - \delta \right\}.$$

Putting Government in

If we collect our expressions we get the following set of equations:

- The government spending condition, where we have taken account of the fact that the revenue was raised in the prior period and deflated by $1 + g$ to get

$$G_1^{1-\nu} = \frac{1}{1+g} \frac{\tau_Y}{\omega} L^{1-\alpha-\nu} K_1^\alpha (1-\psi)^\nu \quad (12)$$

- The resource constraint

$$C_1 = (1 - \tau_Y) L^{1-\alpha-\nu} K_1^\alpha ((1 - \psi) G_1)^\nu - [g + \delta] K_1 \quad (13)$$

- The labor condition

$$L^{\gamma+\alpha+\nu} = \frac{\beta}{1+\tau} \frac{(1-\tau_Y)}{C_1} (1-\alpha-\nu) K_1^\alpha ((1-\psi) G_1)^\nu. \quad (14)$$

- And the capital condition

$$1 = \frac{\beta}{1+g} \left\{ (1-\tau_Y) \alpha L^{1-\alpha-\nu} K_1^{\alpha-1} ((1-\psi) G_1)^\nu + 1 - \delta \right\}. \quad (15)$$

Putting Government in

- This is a two-variable system, since given L and K_1 we can determine G_1 from equation (12).
- Given these three variables, we can determine C_1 from (13).
- Given these four variables, we use conditions (14) and (15) to check that our two original values were correct.
- The derivation of this system also verifies our assumption that there existed a balanced growth path with capital and government spending growing at the same rate as labor productivity.
- BUT we are going to need *fsolve* again since it is a simultaneous set of equations.

Putting Government in

fsolve is a nonlinear equation solver.

```
X0 = [.5 .5 .5]';  
ERR = @(X) ERRB(X,Z,tau,gamma,alpha,kappa,beta);  
% defining function handle to pass in variables.  
ERR(X0) % Produces check value to see how far off guess is  
solution = fsolve(ERR,X0);
```

- We define a subroutine function ERRB which takes in our 3 guess values and returns the equation errors.
- At the solution $ERR(\text{solution}) = [0;0;0]$.
- BUT we need to pass in a bunch of parameters which we did through the function handle by creating ERR which is ERRB with the parameters imbedded.

Putting Government in

- Fsolve uses the **Newton Method** which works as follows. You specify an initial guess x . Then,
- Compute the numerical derivative of f at x , which is given by

$$f'(x) \cong \frac{f(x + \delta) - f(x)}{\delta}$$

for small δ .

- Solve the following linear equation for our new guess value x'

$$f(x) + f'(x)(x' - x) = 0.$$

- If $\|f(x') - 0\| < tol$, quit. Otherwise, go back to step 1 using x' as the initial starting guess.

Putting Government in

Doing it analytically:

Start with the G equation and rearrange it to get:

$$\left[\frac{G_1}{L} \right]^{1-\nu} = \frac{1}{1+g} \frac{\tau_Y}{\omega} \left[\frac{K_1}{L} \right]^\alpha (1-\psi)^\nu \quad (16)$$

Then, rearrange the capital equation to get:

$$1 = \frac{\beta}{1+g} \left\{ (1-\tau_Y)\alpha \left[\frac{K_1}{L} \right]^{\alpha-1} \left((1-\psi) \left[\frac{G_1}{L} \right] \right)^\nu + 1 - \delta \right\}. \quad (17)$$

Plug in our result for G/L to get

$$1 = \frac{\beta}{1+g} \left\{ \begin{array}{c} (1-\tau_Y)\alpha \left[\frac{K_1}{L} \right]^{\alpha-1} \\ \times (1-\psi)^\nu \left[\frac{1}{1+g} \frac{\tau_Y}{\omega} \left[\frac{K_1}{L} \right]^\alpha (1-\psi)^\nu \right]^{\frac{\nu}{1-\nu}} \\ + 1 - \delta \end{array} \right\}. \quad (18)$$

Putting Government in

If I play a bit more with the labor equation

$$\begin{aligned}L^{\gamma+\alpha+\nu} &= \frac{\beta}{1+\tau} \frac{(1-\tau_Y)}{C_1} (1-\alpha-\nu) K_1^\alpha ((1-\psi) G_1)^\nu. \\ \implies L^{1+\gamma} &= \frac{\beta}{1+\tau} \frac{(1-\tau_Y)}{C_1} (1-\alpha-\nu) K_1^\alpha ((1-\psi) G_1)^\nu L^{1-\alpha-\nu} \\ \implies L^{1+\gamma} &= \frac{\beta}{1+\tau} \frac{(1-\tau_Y)}{C_1} (1-\alpha-\nu) Y_1\end{aligned}$$

So, if $(1-\tau_Y) Y_1 / C_1$ is constant, then so is labor.