Midterm 2025-2026

Solutions

November 11, 2025

1 Production (5 points)

Consider the production function :

$$y = \tfrac{1}{2}x_1^{1/2}x_2^{1/2}$$

1.1 Technical rate of substitution (TRS) (2 points)

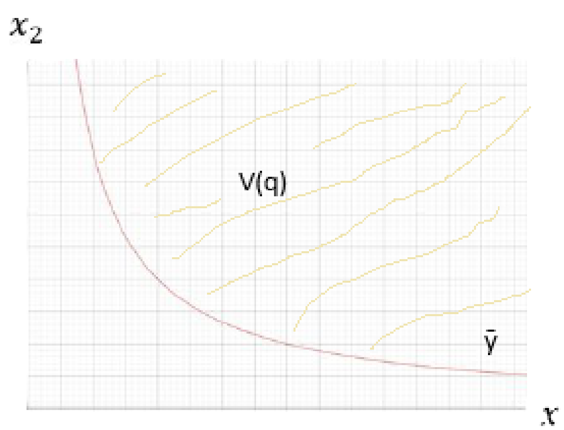
$$\text{TRS} = -\frac{MP_1}{MP_2}$$

$$MP_1 = \frac{\partial y}{\partial x_1} = \tfrac{1}{4}x_2^{1/2}x_1^{-1/2}, \qquad MP_2 = \frac{\partial y}{\partial x_2} = \tfrac{1}{4}x_1^{1/2}x_2^{-1/2}$$

$$\text{TRS} = -\frac{MP_1}{MP_2} = -\frac{x_2}{x_1}$$

1.2 Isoquant and input requirement set (1 point)

Isoquant for output level $\bar{y}$ is a smooth, convex, downward-sloping hyperbola (in red). Input requirement set $V(q)$ is all input combinations on or above (to the northeast of) the isoquant (in yellow).



1.3 TRS interpretation (2 points)

$$\text{TRS}_{12} = -\frac{x_2}{x_1}.$$

From both the TRS form and the graph of the isoquant, one can conclude that:

If x_2 is large and x_1 is small, the TRS (i.e., the slope of the isoquant) is strongly negative. This implies that if one increases x_1 from a small value, one can decrease x_2 by a lot and keep producing the same amount.

In contrast, if x_2 is small and x_1 is large, the TRS is close to zero. This implies that if one increases x_2 from a small value, one can decrease x_1 by a lot and keep producing the same amount.

Economic intuition: if we already use a lot of x_2 (x_1), using more of x_2 (x_1) is not that productive (diminishing marginal productivity). If we already use a lot of x_2 (x_1), we can decrease x_2 by a lot and only increase x_1 (x_2) by a little and keep producing the same output. This relationship reflects a preference for a "balanced" input bundle over an "extreme" input bundle.

2 Profit maximization (8 points)

2.1 WAPM calculations (3.5 points)

WAPM requires that for each observation t the observe profit at prices (p_t, w_t) is at least as large as the profit calculated by evaluating the potential bundle at the same prices:

$$p^t y^t - w^t x^t \ \geq \ p^t y^s - w^t x^s \quad \forall t, t \neq s.$$

<u>At prices $(p^1, w^1) = (6, 3)$:</u>

$$6\cdot8-3\cdot2 \ \geq \ 6\cdot a-3\cdot3 \iff 48-6 \ \geq \ 6a-9 \iff 42 \geq 6a-9 \iff 6a \leq 51 \iff a \leq 8.5.$$

<u>At prices $(p^2, w^2) = (4, 2)$:</u>

$$4\cdot a - 2\cdot 3 \ \geq \ 4\cdot 8 - 2\cdot 2 \iff 4a - 6 \geq 32 - 4 \iff 4a \geq 34 \iff a \geq 8.5.$$

Combining, WAPM holds iff

$$a = 8.5\,.$$

2.2 Explanation (3 points) WAPM is a necessary condition (but not sufficient) because we do not observe all the choices that are feasible. To be necessary and sufficient one would have to show that it holds for all production plans.

Since it is necessary but not sufficient, if we reject it we can conclude the firm is did not maximize profits. To be necessary However, if we do not reject the WAPM, we can only say that the firms' behavior is consistent with profit maximization, but not that the firm is profit maximizing. It could still be that there are some plans that violate the WAPM but the firms has not yet selected them.

2.3 Differentiable case (1.5 points) First we can substitute the production function $y = f(x)$, to get:

$$\pi(p, w) = pf(x) - wx$$

The FOC for x_1 is

$$p\frac{\partial f(x)}{\partial x_1} - wx_1 = 0$$

Now consider:

$$p\frac{\partial f(x)}{\partial x_1} - wx_1 > 0$$

A small increase in x_1 raises profit. Keep increasing x_1 until $p\frac{\partial f(x)}{\partial x_1} = wx_1$, where the value of the marginal product equals its cost.

3 Constrained Optimization (4 points)

3.1 We want to minimize costs

$$c = 4x_1 + x_2$$

subject to the production constraint

$$x_1^{1/3}x_2^{2/3} = 12.$$

Set up the Lagrangian:

$$\mathcal{L} = 4x_1 + x_2 - \lambda\left(x_1^{1/3}x_2^{2/3} - 12\right).$$

Take FOCs:

$$\frac{\partial \mathcal{L}}{\partial x_1} : 4 - \lambda \cdot \frac{1}{3}x_1^{-2/3}x_2^{2/3} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_2} : 1 - \lambda \cdot \frac{2}{3}x_1^{1/3}x_2^{-1/3} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} : x_1^{1/3}x_2^{2/3} - 12 = 0.$$

Divide the first FOC by the second:

$$\frac{4}{1} = \frac{\frac{1}{3}\lambda x_1^{-2/3}x_2^{2/3}}{\frac{2}{3}\lambda x_1^{1/3}x_2^{-1/3}} \iff 4 = \frac{1}{2} \cdot \frac{x_2}{x_1}.$$

It follows: $x_2 = 8x_1$.

Substitute into the constraint:

$$x_1^{1/3}(8x_1)^{2/3} = 12,$$

$$x_1^{1/3} \cdot 8^{2/3} \cdot x_1^{2/3} = 12,$$
$$4x_1 = 12 \quad \Rightarrow \quad x_1^* = 3 \quad \text{and} \quad x_2^* = 24.$$

Finally, we can find the minimum costs by plugging the factor demands into the cost function:

$$c = 4x_1^* + x_2^* \iff c = 4 \cdot 3 + 24 \iff c = 36.$$

4 Cost Functions (3 points)

5.1 Show that SMC = SAC at the minimum of SAC.

Let y^* be the output level where SAC is minimized. We want to show that

$$\frac{\partial c(y^*)}{\partial y} = \frac{c(y^*)}{y^*}.$$

Note: $c(y) = c(\omega, y, \bar{x})$ is the short–run cost function, but we suppress ω and $\bar{x}$ for notation.

Average cost:

$$SAC(y) = \frac{c(y)}{y}.$$

Take the derivative wrt y:

$$\frac{\partial SAC}{\partial y} = \frac{yc'(y) - c(y)}{y^2}.$$

At the minimum y^* we have

$$\frac{y^* c'(y^*) - c(y^*)}{(y^*)^2} = 0 \qquad \Longrightarrow \qquad c'(y^*) = \frac{c(y^*)}{y^*}.$$

Thus, at y^*:

$$SMC(y^*) = \frac{c(y^*)}{y^*} = SAC(y^*)$$

which was to be shown.