# Formulas ALM

Definition of yield

$$P(t) = e^{-yt}; \;\; y(t) = -\frac{1}{t} ln\left(P(t)\right)$$

Forward rates

$$y_F(t) = y(t) + t \cdot y'(t)$$

Annual compounding

$$i = e^y - 1$$

Average bond yield

$$B = B(\bar{y}) \stackrel{!}{=} \sum_{i=1}^{n} e^{-\bar{y}t_i c(t_i)}$$

## Present Value sensitivity

Present value of the cash flow

$$B(y) = \int_0^\infty e^{-y(t)t} dC(t)$$

Derivatives of the PV of the cash flow

$$B'(y) = -\int_0^\infty t e^{-y(t)t} dC(t), \; B''(y) = -\int_0^\infty t^2 e^{-y(t)t} dC(t)$$

Taylor approximation of the change in PV due to a shift in the yield curve

$$B(y + \Delta\bar{y}) - B(y) \approx B'(y)\Delta\bar{y} + \frac{1}{2} B''(y)(\Delta\bar{y})^2$$

Duration and convexity

$$D = D(y) = -B'(y)/B(y)$$

$$C = C(y) = B''(y)/B(y)$$

Dollar duration and dollar convexity of the surplus:

$$\begin{aligned} DD \quad &= PV_A \cdot D_A - PV_L \cdot D_L \\ DC \quad &= PV_A \cdot C_A - PV_L \cdot C_L \end{aligned}$$

First order matching

$$D_A = D_L$$

Second order matching

$$PV_A \cdot C_A \geq PV_L \cdot C_L$$

## Term structure equilibrium models

Vasicek model

$$dr(t) = a(b - r(t))dt + \sigma dz(t)$$

Cox-Ingersoll-Ross model

$$dr(t) = a(b - r(t))dt + \sigma\sqrt{r(t)}dz(t)$$

## Affine term structure

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

Vasicek model

$$B(t,T) = \frac{1 - e^{-a(t-T)}}{a}$$

$$A(t,T) = e^{\frac{(B(t,T)-(T-t))(a^2b-\frac{1}{2}\sigma^2)}{a^2}} \cdot e^{-\frac{\sigma^2 B^2(t,T)}{4a}}$$

CIR model

$$A(t,T) = \left(\frac{2\gamma e^{(\gamma+a)(T-t)/2)}}{(\gamma+a)(e^{\gamma(T-t)}-1)+2\gamma}\right)^{2ab/\sigma^2}$$

$$B(t,T) = \frac{2(e^{(\gamma(T-t)}-1)}{(\gamma+a)(e^{\gamma(T-t)}-1)+2\gamma}$$

$$\gamma = \sqrt{a^2 + 2\sigma^2}$$

## Mean Variance Analysis

Minimum variance Portfolio

$$\min_{\mathbf{w}} \mathbf{w}'\Sigma\mathbf{w},\ \text{subject to (only) } \mathbf{w}'\mathbf{1} = 1$$

Optimal Portfolio of Risky Assets

$$\min_{\mathbf{w}} \mathbf{w}'\Sigma\mathbf{w},\ \text{subject to } \mathbf{w}'\mu = \text{r and (of course) } \mathbf{w}'\mathbf{1} = 1$$

Optimal portfolio with a risk-free asset

$$\min_{w_0,\mathbf{w}} \mathbf{w}'\Sigma\mathbf{w},\ \text{ subject to } \ w_0\mu_0 + \mathbf{w}'\mu = r \ \text{ and } \ w_0 + \mathbf{w}'\mathbf{1} = 1$$