LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

Operational Research

Filipe Rodrigues

frodrigues@iseg.ulisboa.pt

Office 302 Quelhas 2

2025 / 2026

Evaluation Process

Regular period

- Written **exam** (scaled from 0 to 20)

15th May 2026

Repeat period

To get approval (or to improve the grade)

- Written **exam** (scaled from 0 to 20)

2nd June 2026

To get approval in the OR course, the *Final Grade* must be greater than or equal to 9.5.

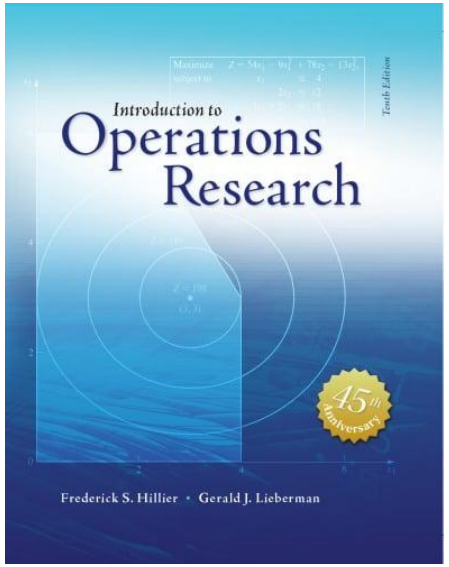
For both periods: An oral examination may be required when the final grade is higher than 17.

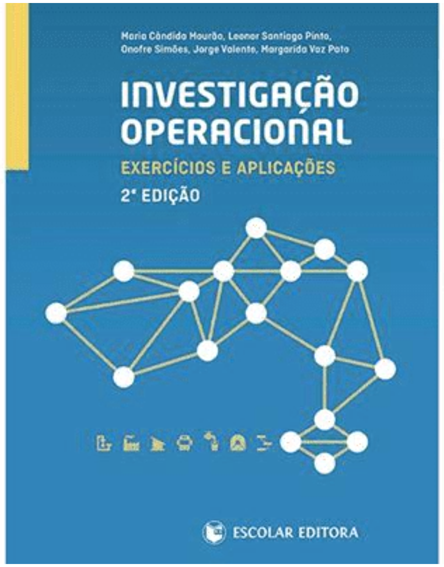
Bibliography

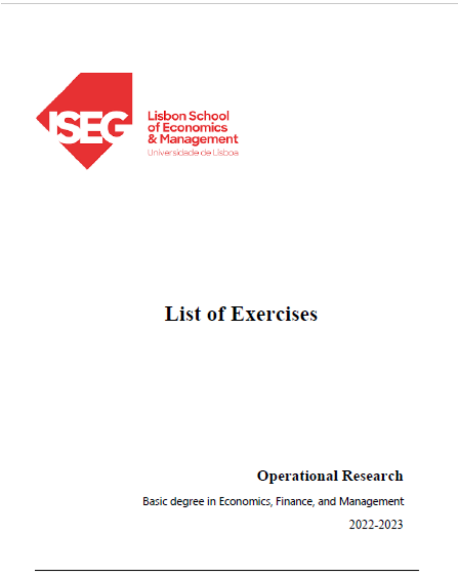
[H&L] F.S. Hillier, G.J. Lieberman, *Introduction to Operations Research*, 10th edition, McGraw-Hill, International Edition, New York, 2015

[Mourão *et. al*] M.C. Mourão, L. Santiago Pinto, O. Simões, J. Valente, M.V. Pato, *Investigação Operacional: Exercícios e Aplicações*, 2nd edition, Escolar Editora, Lisboa, 2019 (Written in Portuguese!)

[Exercises] Operational Research, List of Exercises.







If you have doubts:

- Come to my office
  (but tell us first …)

- Online by TEAMS (or by e-mail)
  (any time)

Second

World War → Operational Research → Industry

Operational Research

(Around 1950)

What is Operational Research?

Consists of studying operations (activities) within an organization in order to make them more efficient.

↓

Decrease costs and increase profits

What is Operational Research?

OPERATIONAL RESEARCH IS COMPOSED OF THE FOLLOWING ACTIVITIES:

1. DEFINE THE PROBLEM
   Identify decisions, data and objectives
2. BUILD A MODEL FOR THE PROBLEM
   Create a mathematical representation of the problem
3. USE SOLUTION METHODS
   Create/apply scientific procedures for solving the model created
4. MODEL VALIDATION
   Test and adjust the mathematical model and the solution methods
5. OBTAIN SOLUTIONS
   Propose solutions to the decision-maker
6. IMPLEMENTATION
   Implementation of the designed procedures and solutions in practice

■ **TABLE 1.1** Applications of operations research to be described in application vignettes

Organization	Area of Application	Section	Annual Savings
Federal Express	Logistical planning of shipments	1.3	Not estimated
Continental Airlines	Reassign crews to flights when schedule disruptions occur	2.2	$40 million
Swift & Company	Improve sales and manufacturing performance	3.1	$12 million
Memorial Sloan-Kettering Cancer Center	Design of radiation therapy	3.4	$459 million
United Airlines	Plan employee work schedules at airports and reservations offices	3.4	$6 million
Welch's	Optimize use and movement of raw materials	3.3	$150,000
Samsung Electronics	Reduce manufacturing times and inventory levels	4.3	$200 million more revenue
Pacific Lumber Company	Long-term forest ecosystem management	6.7	$398 million NPV
Procter & Gamble	Redesign the production and distribution system	8.1	$200 million
Canadian Pacific Railway	Plan routing of rail freight	9.3	$100 million
United Airlines	Reassign airplanes to flights when disruptions occur	9.6	Not estimated
U.S. Military	Logistical planning of Operations Desert Storm	10.3	Not estimated
Air New Zealand	Airline crew scheduling	11.2	$6.7 million
Taco Bell	Plan employee work schedules at restaurants	11.5	$13 million
Waste Management	Develop a route-management system for trash collection and disposal	11.7	$100 million

CHAPTER 1.

LINEAR PROGRAMMING

Summary:

- Formulate and interpret LPP;
- Assumptions, properties, and main definitions in LP;
- Solve an LPP with 2 variables either by evaluating all corner points or by the graphical method (all possible cases);
- Solve LPPs by using the Excel Solver.

*See H&L, Chapter 3, pages 25 - 71

Linear Programming problems

A linear programming problem (LPP) has the general form:

$$\min/\max \sum_{j=1}^{n} c_j x_j$$ ← Objective function (OF)

$$
\begin{aligned}
s.t. \quad & \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i, && i = 1, \dots, k \\
& \sum_{j=1}^{n} a_{ij}\, x_j \geq b_i, && i = k+1, \dots, \ell \\
& \sum_{j=1}^{n} a_{ij}\, x_j = b_i, && i = \ell+1, \dots, m
\end{aligned}
$$
← Functional constraints

$$
\begin{aligned}
& x_j \geq 0 && j = 1, \dots, s \\
& x_j \leq 0 && j = s+1, \dots, t \\
& x_j \ free && j = t+1, \dots, n
\end{aligned}
$$
← Signal constraints

where:

- x_j → decision variable
- c_j → objective function coefficient of variable x_j
- b_i → right-hand side coefficient of constraint i
- a_{ij} → technical coefficient of variable x_j in constraint i

LP1. A small farmer produces packs of strawberry and banana-flavored milk and has a profit equal to 20 and 30 cents per each produced pack.
The farmer has resources for producing only 30 packs of milk and must ensure that the number of banana-flavored packs is at least twice the number of strawberry-flavored packs.
How many packs of each type should be produced to achieve the highest possible profit?

Exercise 1

Alfredo has a farm where he wants to raise chickens, rabbits, and goats. The price of each chicken, rabbit, and goat is 2, 5, and 40m.u., respectively.

To receive a financial support to the farm, the sum of the number of legs of all animals on the farm cannot be less than 30 and the sum of animal heads cannot be less than 15. In addition, the number of chickens cannot exceed 20% of the number of the remaining animals and the farm only has capacity for feeding up to 800 animals.

It is estimated to obtain a profit of 1, 2, and 30, m.u. for each chicken. rabbit, and goat and Alfredo wants to obtain a profit not lower than 500 m.u.

The chicken's house is small and therefore can only accommodate up to 20 chickens.

There is a large stable on the farm reserved to the goats and rabbits. In this stable there are 500 compartments, and each compartment can be empty or (when occupied) must contain exactly one goat and two rabbits, because the goats are afraid of being alone at night. There is no other place available for the goats on the farm, but there is an extra compartment with capacity for at most 50 rabbits.

Formulate the problem to determine the number of animals of each type that Alfredo should buy for his farm to minimize the total purchase cost of the animals (considering all constraints of the problem).

- **Proportionality**: The contribution of each variable to the value of the objective function and to the left-hand side of the constraints is proportional to the value of such a variable.

$4x_1$ ✗ $3x_1^2$ ✗ $\sqrt{x_2}$ ✗ e^{x_1}

- **Additivity**: The value of the objective function and the value of the left-hand-side of the constraints are the sum of the individual contributions of the decision variables.

$4x_1 + x_2$ ✗ $3x_1x_2$ ✗ x_1/x_2

- **Divisibility:** The decision variables assume real values ($x_j \in R$).

- **Certainty:** Every coefficient/parameter (c_j, a_{ij}, and b_i) is assumed to be a known constant.

LP definitions

- The solution of a LPP is represented by a vector $x = (x_1, \dots, x_n)$
- The set of constraints of an LPP defines a region called the **Feasible Region** (FR)
- The corner points in the FR are called **extreme points**
- Classification of solutions:
  - **Feasible solution** (FS) → belongs to the FR
  - **Infeasible / Non-feasible solution (NFS)** → does not belong to the FR (does not satisfy at least one of the constraints).
  - **Optimal solution** → FS with the best objective function value
  - **Alternative optimal solution** → FS with an objective function value equal to the best possible objective function value.
- The **optimal value** of an LPP is the value of the objective function at any optimal solution.
- A constraint is **binding** in a feasible solution if it holds on the equality on that solution.

Prop 1. The Feasible Region of an LPP is either an empty set or a convex set.

Prop 2. If the Feasible Region of an LPP is nonempty and bounded, then at least one optimal solution exists.

Prop 3. If an LPP has an optimum, then at least one of its extreme points is an optimal solution.

Prop 4. Given an LPP with an optimum, if an extreme point has no adjacent extreme points with a better objective function value, then that point is an optimal solution.

Method for solving LPPs by evaluating the extreme points

1. Represent the feasible region of the problem.

2. If the feasible region is non-empty and bounded, determine all extreme points.

3. Determine the objective function value of each extreme point. The point (or points) with the best objective function value is the optimal solution of the problem and the associated value is the optimal value.

Graphical method

1. Represent the feasible region
2. If the feasible region is empty

   STOP - The problem is infeasible.

3. Else

   3.1. Represent the gradient vector of the objective function

   3.2. Draw a line perpendicular to the gradient

   The equation of such a line is $c_1x_1 + c_2x_2 = k$, for $k \in \mathbb{R}$

   3.3. For a maximization problem, move the line in the direction of the gradient.
   For a minimization problem, move the line in the opposite direction of the gradient.

   3.4. If the line never leaves the feasible region

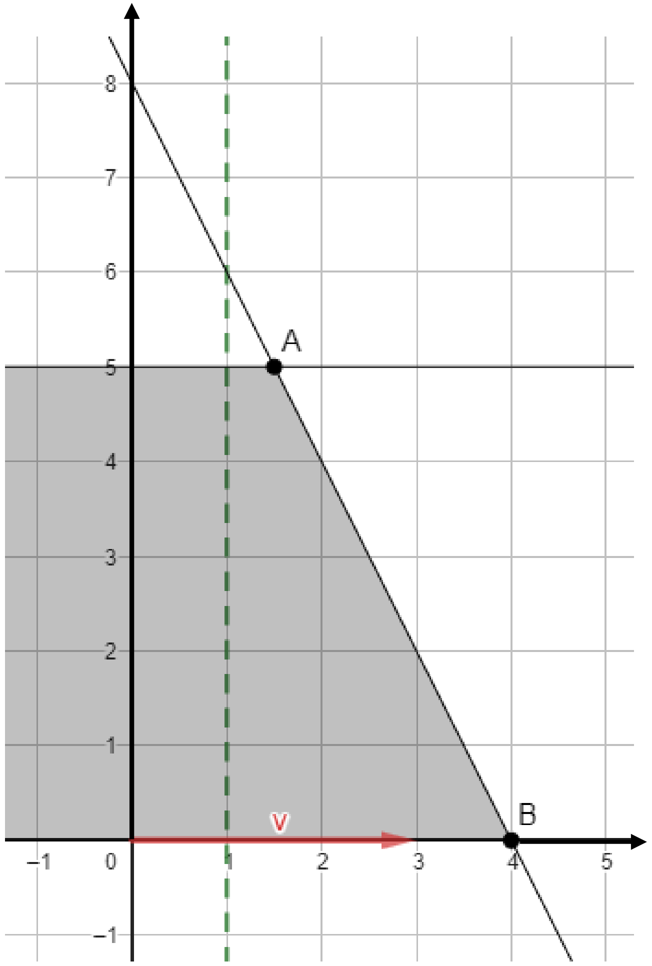
   The problem is unbonded.

   3.5. Else

   The optimal solution is the last point (or set of points) intersected by the line before leaving the feasible region.

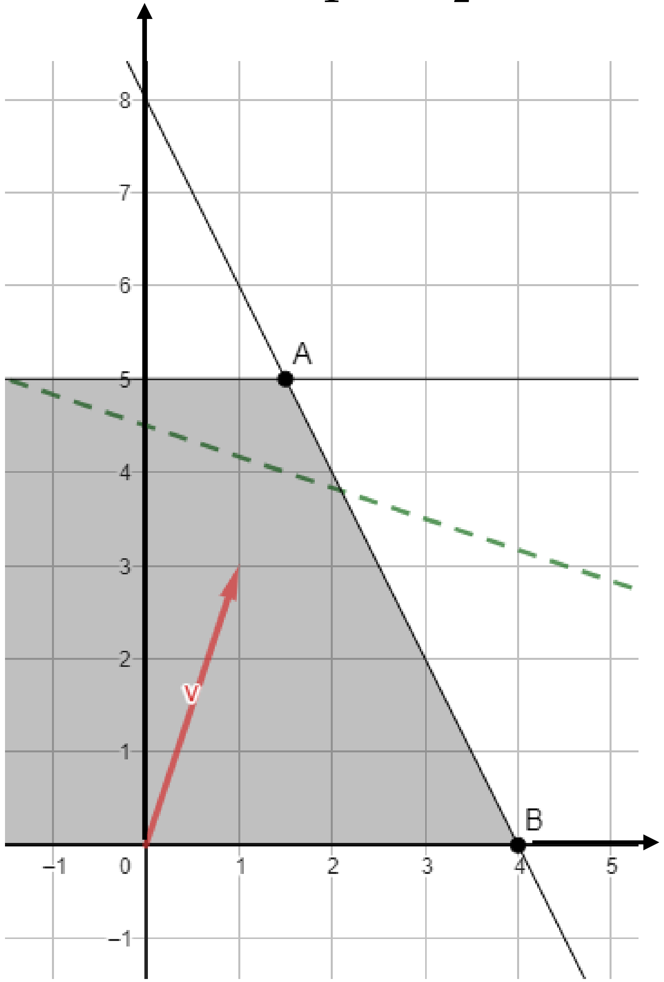
Graphical method - Example

$\min 3x_1$



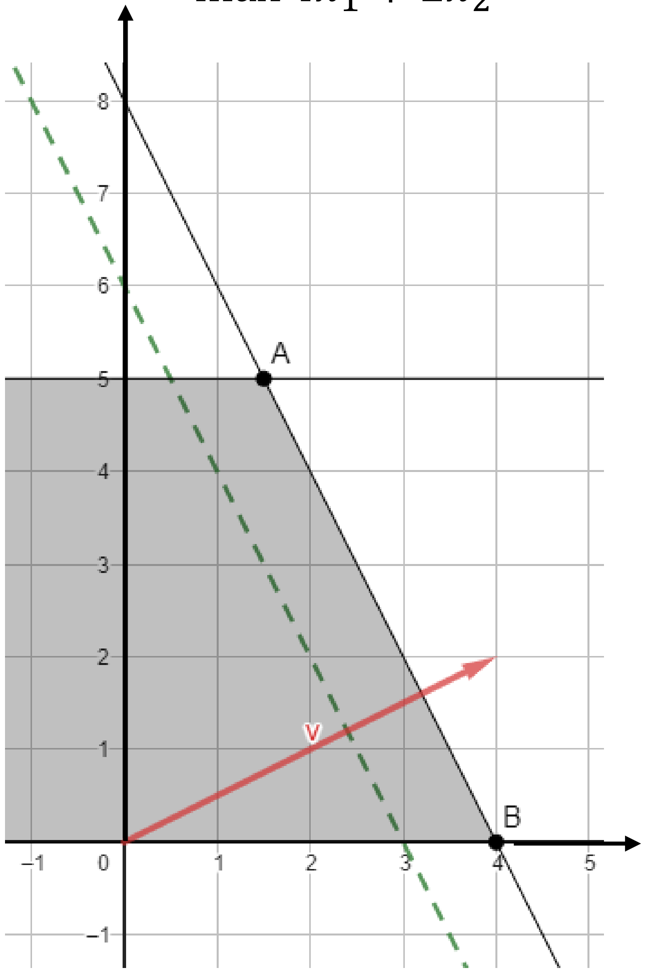
Unbounded problem

$\max x_1 + 3x_2$



The optimal solution is point A

$\max 4x_1 + 2x_2$



All points in the semi-line AB are optimal solutions

Solve an LPP in the Excel spreadsheet

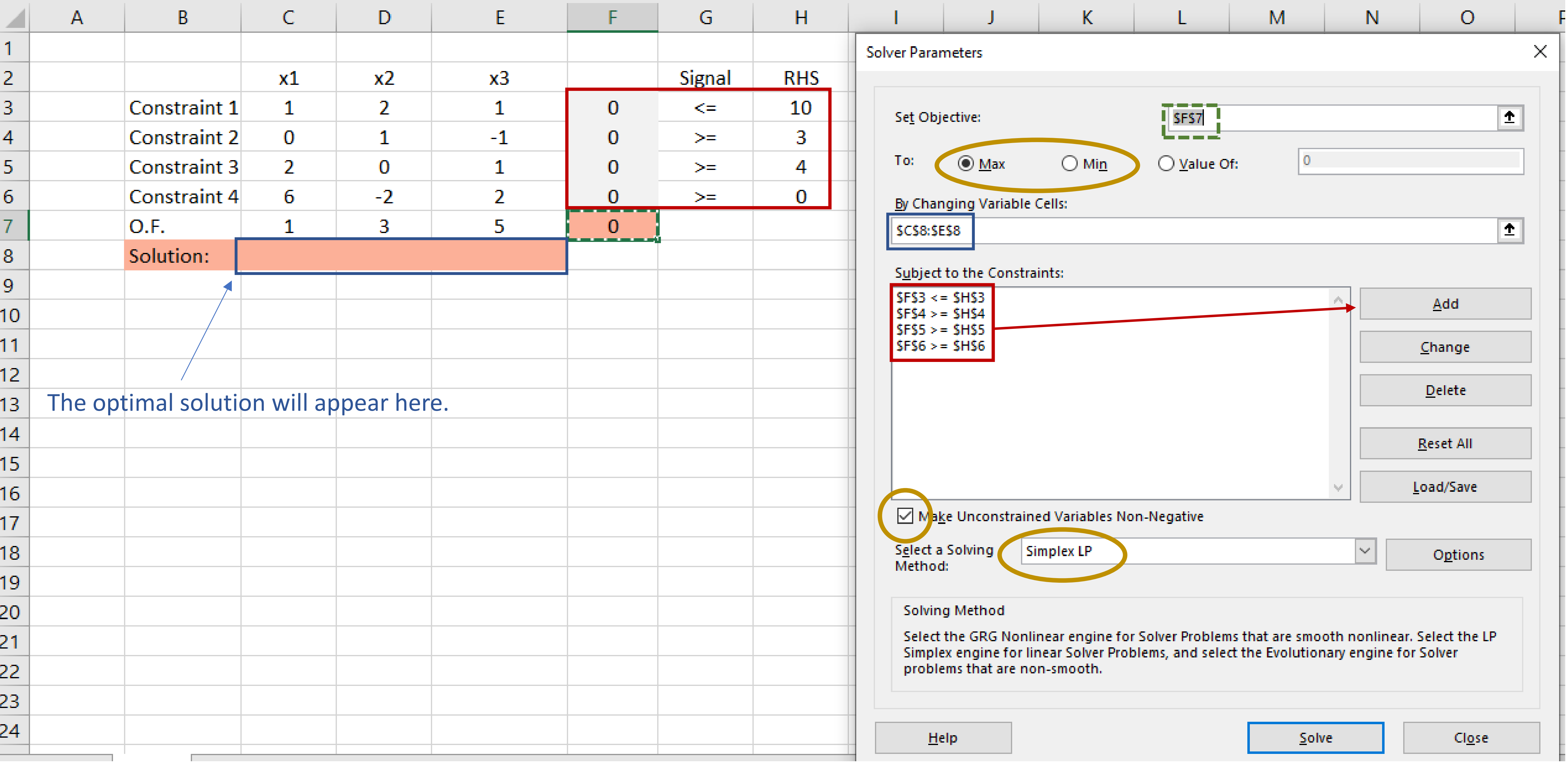
$$\begin{aligned} \max \quad & x_1 + 3x_2 + 5x_3 \\ s.t. \quad & x_1 + 2x_2 + x_3 \leq 10 \\ & x_2 - x_3 \geq 3 \\ & 2x_1 + x_3 \geq 4 \\ & 6x_1 - 2x_2 + 2x_3 \geq 0 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

F
=SUMPRODUCT(C3:E3;$C$8:$E$8)
=SUMPRODUCT(C4:E4;$C$8:$E$8)
=SUMPRODUCT(C5:E5;$C$8:$E$8)
=SUMPRODUCT(C6:E6;$C$8:$E$8)
=SUMPRODUCT(C7:E7;$C$8:$E$8)

	A	B	C	D	E	F	G	H
1								
2			x1	x2	x3		Signal	RHS
3		Constraint 1	1	2	1	0	<=	10
4		Constraint 2	0	1	-1	0	>=	3
5		Constraint 3	2	0	1	0	>=	4
6		Constraint 4	6	-2	2	0	>=	0
7		O.F.	1	3	5	0		
8		Solution:						
9								

How to make the Solver available in the tab *Data* of the Excel: **Windows:** File / Options / Add-ins / Go / Solver Add-in **Mac:** Tools / Add-ins / Solver Add-in

Solve an LPP in the Excel spreadsheet



LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

CHAPTER 2.

Simplex Method

Summary:

- Write an LPP in the standard form and in the augmented form;
- Identify Basic solutions of LPPs and their properties;
- Solve an LPP by using the simplex method (all possible cases);
- Understand the ideas behind the simplex method.

*See H&L, Chapter 4, pages 93 - 115

The ***standard form*** and the ***augmented form*** of a maximization LPP are as follows:

Standard form

$$\max \sum_{j=1}^{n} c_j x_j$$

$$s.t. \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i, \qquad i = 1, \dots, m$$

$$x_j \geq 0, \qquad j = 1, \dots, n$$

(Constraints with ≤ and variables ≥ 0)

Augmented form

$$\max \sum_{j=1}^{n} c_j x_j$$

$$s.t. \sum_{j=1}^{n} a_{ij}\, x_j + s_i = b_i, \qquad i = 1, \dots, m$$

$$x_j \geq 0, \qquad j = 1, \dots, n$$

$$s_i \geq 0, \qquad i = 1, \dots, m$$

(Constraints with = and variables ≥ 0)

To write a general LPP in the augmented form, start by writing it in the standard form and then add positive ***slack variables*** s_i to convert the $\leq$ constraints into equalities.

The standard form of an LPP

Any LPP can be written in the standard form of a maximization problem:

- **Minimization problem**

  A minimization problem can be converted into a maximization problem by multiplying the o.f. by -1.

$$min \;\; f(x_1, \dots, x_n) = \sum_{j=1}^{n} c_j x_j \qquad \Leftrightarrow \qquad max \; - f(x_1, \dots, x_n) = \sum_{j=1}^{n} -c_j x_j$$

Example:

$$min\; z = \; 3x_1 - 2x_2 + x_3 \qquad \Leftrightarrow \qquad max\; -z = -3x_1 + 2x_2 - x_3$$

- **Constraints $\geq$**

  A " $\geq$ " constraint can be converted into a " $\leq$ " constraint by multiplying it by -1.

$$\sum_{j=1}^{n} a_{ij}\, x_j \geq b_i \qquad \Leftrightarrow \qquad \sum_{j=1}^{n} -a_{ij}\, x_j \leq -b_i$$

Example:

$$3x_1 - 2x_2 + x_3 \geq -5 \qquad \Leftrightarrow \qquad -3x_1 + 2x_2 - x_3 \leq 5$$

The standard form of an LPP

❑ Equality constraints

An equality constraint can be converted into two " $\leq$ " constraints.

$$\sum_{j=1}^{n} a_{ij}\, x_j = b_i \quad \Leftrightarrow \quad \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i \quad and \quad \sum_{j=1}^{n} -a_{ij}\, x_j \leq -b_i$$

Example:

$$3x_1 - 2x_2 + x_3 = 5 \quad \Leftrightarrow \quad 3x_1 - 2x_2 + x_3 \leq 5 \quad and \quad -3x_1 + 2x_2 - x_3 \leq -5$$

❑ Variables $\leq$ 0 or *free*

Variables ≤ 0 or *free* can be equivalently replaced by new variables ≥ 0.

$$x_j \leq 0 \quad \Leftrightarrow \quad x_j = -\overline{x_j} \quad with \quad \overline{x_j} \geq 0$$

$$x_j\ free \quad \Leftrightarrow \quad x_j = x_j^+ - x_j^- \quad with \quad x_j^+, x_j^- \geq 0$$

Standard form

$$\max \sum_{j=1}^{n} c_j x_j$$

$$s.t. \sum_{j=1}^{n} a_{ij}\, x_j \leq b_i, \qquad i = 1, \dots, m$$

$$x_j \geq 0, \qquad j = 1, \dots, n$$

Solution: $(x_1^*, \dots, x_n^*)$

Points resulting from the intersection of two constraints

(Feasible) Corner Points of the FR

Infeasible Corner Points (outside the FR)

Augmented form

$$\max \sum_{j=1}^{n} c_j x_j$$

$$s.t. \sum_{j=1}^{n} a_{ij}\, x_j + s_i = b_i, \qquad i = 1, \dots, m$$

$$x_j \geq 0, \qquad j = 1, \dots, n$$

$$s_i \geq 0, \qquad i = 1, \dots, m$$

Solution: $(x_1^*, \dots, x_n^*, s_1^*, \dots, s_m^*)$

Basic Solutions

Basic Feasible Solutions

Basic Non-Feasible Solutions

Basic solutions of an LPP

Consider the following LPP with ***m*** constraints and ***n+m*** variables written in the augmented form:

$$\max \quad \sum_{j=1}^{n} c_j x_j$$

$$s.t. \quad \sum_{j=1}^{n} a_{ij}\, x_j + s_i = b_i, \qquad i = 1, \dots, m$$

$$x_j \geq 0, \qquad j = 1, \dots, n$$

$$s_i \geq 0, \qquad i = 1, \dots, m$$

In a **basic solution** $(x_1, \dots, x_n, s_1, \dots, s_m)$, each variable is designated as **non-basic variable** or as **basic variable** and:

- The number of basic variables equals the number of functional constraints (***m***).
- The number of non-basic variables equals the total number of main variables (***n***).
- All non-basic variables are equal to zero.
- The set of basic variables is called the **basis** of the solution.

Basic solutions of an LPP

To identify a basic solution of an LPP with ***m*** constraints and ***n+m*** variables written in the augmented form:

1. Set ***n*** variables equal to zero (which will be the **non-basic variables**)
2. Solve the system of equations

$$\sum_{j=1}^{n} a_{ij}\, x_j + s_i = b_i, \qquad i = 1, \dots, m$$

to determine the value of the remaining ***m*** variables (which will be the **basic variables**).

3. If the system has a unique solution:

   3.1 The obtained solution - (non-basic variables + basic variables) - is a **basic solution**.

   3.2 If such a solution satisfies all the signal constraints

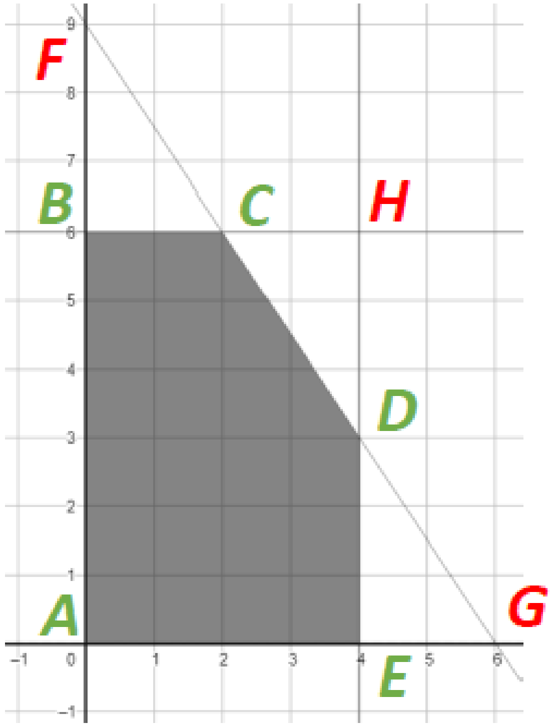
$$x_j \geq 0, \qquad j = 1, \dots, n$$
$$s_i \geq 0, \qquad i = 1, \dots, m$$

It is a **basic feasible solution** (BFS). Otherwise, it is a **basic non-feasible solution** (BNFS).

Example: Identify all the basic solutions of the following LPP

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1, x_2 \ge 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + s_1 = 4 \\ & 2x_2 + s_2 = 12 \\ & 3x_1 + 2x_2 + s_3 = 18 \\ & x_1, x_2, s_1, s_2, s_3 \ge 0 \end{aligned}$$

x_1	x_2	s_1	s_2	s_3		
0	0	4	12	18	BFS	A
0	-	0	-	-		
0	6	4	0	6	BFS	B
0	9	4	-6	0	BNFS	F
4	0	0	12	6	BFS	E
-	0	-	0	-		
6	0	-2	12	0	BNFS	G
4	6	0	0	-6	BNFS	H
4	3	0	6	0	BFS	D
2	6	2	0	0	BFS	C



Adjacent BFSs

- Each BFS corresponds to a corner point of the feasible region.

- Two BS are adjacent if their set of non-basic variables differs in exactly one variable.

Example: In the previous example the BS associated with the corner points B and C are adjacent

Point B ⟶ BFS: (0, 6, 4, 0, 6) ⟶ Non-basic variables $\{x_1, s_2\}$

Point C ⟶ BFS: (2, 6, 2, 0, 0) ⟶ Non-basic variables $\{s_3, s_2\}$

- Non-basic variables always take value zero, but variables with value zero are not necessarily non-basic variables.

- The simplex method is an iterative method used to determine the optimal solution of an LPP.

- Its starts from an initial BFS, then successively goes through adjacent BFSs until determine the optimal one or to prove that it does not exist.

> At each iteration, a basic variable in the current BFS becomes non-basic and a non-basic variable becomes a basic variable

- BFS can easily be identified by performing elementary operations with the functional constraints of the model in the augmented form.

Original Problem

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 \le 4 \\ & 2x_2 \le 12 \\ & 3x_1 + 2x_2 \le 18 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Augmented form

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + s_1 = 4 \\ & 2x_2 + s_2 = 12 \\ & 3x_1 + 2x_2 + s_3 = 18 \\ & x_1, x_2, s_1, s_2, s_3 \ge 0 \end{aligned}$$

BFS $(0, 0, 4, 12, 18)$

$$R_3^{new} = R_3^{old} - R_2^{old}$$

$$R_2^{new} = \frac{1}{2} R_2^{old}$$

Reformulation

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + s_1 = 4 \\ & x_2 + \frac{1}{2} s_2 = 6 \\ & 3x_1 - s_2 + s_3 = 6 \\ & x_1, x_2, s_1, s_2, s_3 \ge 0 \end{aligned}$$

BFS $(0, 6, 4, 0, 6)$

Original Problem

$$
\begin{aligned}
\text{max} \quad & z = 3x_1 + 5x_2 \\
\text{s.t.} \quad & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Augmented form

$$
\begin{aligned}
\text{max} \quad & z - 3x_1 - 5x_2 = 0 \\
\text{s.t.} \quad & x_1 + s_1 = 4 \\
& 2x_2 + s_2 = 12 \\
& 3x_1 + 2x_2 + s_3 = 18 \\
& x_1, x_2, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

Initial tableau

BV	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-3	-5	0	0	0	0
s_1	0	1	0	1	0	0	4
s_2	0	0	2	0	1	0	12
s_3	0	3	2	0	0	1	18

Simplex Method – Update the simplex tableau

Initial tableau

BV	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-3	-5	0	0	0	0
s_1	0	1	0	1	0	0	4
s_2	0	0	2	0	1	0	12
s_3	0	3	2	0	0	1	18

- BFS $(0, 0, 4, 12, 18)$
- $EC: x_2$ goes to the basis
- $LC: \min\left\{\frac{12}{2}, \frac{18}{3}\right\} = \frac{12}{2}$ $\rightarrow s_2$ leaves the basis

<u>Elementary Operations:</u>

$$R_0^{new} = R_0^{old} + \frac{5}{2} R_2^{old}$$

$$R_1^{new} = R_1^{old}$$

$$R_2^{new} = \frac{1}{2} R_2^{old}$$

$$R_3^{new} = R_3^{old} - R_2^{old}$$

Tableau #1

BV	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-3	0	0	5/2	0	30
s_1	0	1	0	1	0	0	4
x_2	0	0	1	0	1/2	0	6
s_3	0	3	0	0	-1	1	6

- BFS $(0, 6, 4, 0, 6)$

Tableau #1

BV	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	-3	0	0	5/2	0	30
s_1	0	1	0	1	0	0	4
x_2	0	0	1	0	1/2	0	6
s_3	0	3	0	0	-1	1	6

- BFS $(0, 6, 4, 0, 6)$
- $EC: x_1$ goes to the basis
- $LC: \min\left\{\frac{4}{1}, \frac{6}{3}\right\} = \frac{6}{3}$ $\rightarrow s_3$ leaves the basis

Elementary Operations:

$$R_0^{new} = R_0^{old} + R_3^{old}$$

$$R_1^{new} = R_1^{old} - \frac{1}{3}R_3^{old}$$

$$R_2^{new} = R_2^{old}$$

$$R_3^{new} = \frac{1}{3}R_3^{old}$$

Tableau #2

BV	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	3/2	1	36
s_1	0	0	0	1	1/3	-1/3	2
x_2	0	0	1	0	1/2	0	6
x_1	0	1	0	0	-1/3	1/3	2

- BFS $(2, 6, 2, 0, 0)$ $\longleftarrow$ Optimal solution
  $z^* = 36$ $\longleftarrow$ Optimal value

Unbounded problem

If there is at least one variable in the z-row with a negative coefficient, such a variable may enter the basis because it improves the objective function value. However, if in the pivotal column there are no positive coefficients, then the new variable entering the basis will not be bounded and therefore it may increase as much as we want, meaning that the problem is unbounded.

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	-2	1	0	6
x_4	0	0	0	1	1	2
x_1	0	1	-1	0	0	1

Alternative optimal solutions

If a non-basic variable has coefficient zero in the z-row of the optimal tableau, then such a variable can enter the basis keeping the objective function value unchanged. This means that the problem has alternative optimal solutions.

BV	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	1	0	6
x_4	0	0	3	1	1	2
x_1	0	1	1	0	0	4

If there is at least one positive coefficient in the column associated with the non-basic variable with coefficient zero in the z-row, then we can determine an alternative optimal solution by performing an extra iteration of the simplex Method.

Step 1: Write the problem in the augmented form.

Step 2: Build the initial tableau.

Step 3: Check if any variable has a negative coefficient in the z-row.

If not, then the current solution is optimal. If there is a non-basic variable with coefficient zero in the z-row, it means that there are alternative optimal solutions.

If yes, choose the variable with the most negative coefficient to enter in the basis (Entering Criterion). The column associated with such a variable is called the pivotal column. Check if any coefficient in the pivotal column is positive.

If not, then the problem is unbounded.

If yes, perform the minimum ratio test to determine the variable that must leave the basis. (Leaving criterion).

Denoting by $v_1, \ldots, v_n$ the positive values in the pivotal column, we compute $\min\left\{\frac{RHS_i}{v_i} \mid v_i > 0\right\}$

Remove the basic variable determined with the minimum ratio test from the basis and update the tableau:

- Divide the pivotal row by the pivot.
- Perform elementary operations in the remaining rows.

Step 4: Go back to Step 3.

- Possible draws happening when applying the leaving criterion or the entering criterion are solved by arbitrary choices.

- In this course, we just solve LPPs that can be written in the augmented form satisfying the following conditions:
  - There is a slack variable in each constraint with a positive signal.
  - The right-hand side of each constraint is a non-negative value.

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq 4 \\ & 7x_1 + 2x_2 \geq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \max \quad & 2x_1 + 5x_2 \\ \text{s.t.} \quad & -x_1 + x_2 + s_1 = 4 \\ & 7x_1 + 2x_2 - s_2 = 18 \Leftrightarrow -7x_1 - 2x_2 + s_2 = -18 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

CHAPTER 3.

Duality and Sensitivity Analysis

Summary:

- Build the dual problem;
- Properties of duality theory;
- Determine the solution of the dual problem by solving the primal problem first (four methods);
- Economic interpretation of the solution of an LPP;
- Determine sensitivity intervals (graphically or by the solver reports);
- Analyze outputs from the Excel solver and perform sensitivity analysis.

*See H&L, Chapters 6 and 7, pages 197 – 253

Any LPP (called primal) has a complementary LPP (called dual) associated with it. A pair of dual problems in the standard form is as follows:

Maximization Problem (Standard form)

$$\max \sum_{j=1}^{n} c_j x_j$$

$$s.t. \sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \qquad i = 1, \dots, m$$

$$x_j \ge 0, \qquad j = 1, \dots, n$$

$$\max \; c^T x$$

$$s.t. \; Ax \le b$$

$$x \ge 0$$

Minimization problem (Standard form)

$$\min \sum_{i=1}^{m} b_i y_i$$

$$s.t. \sum_{i=1}^{m} a_{ji}\, y_i \ge c_j, \qquad j = 1, \dots, n$$

$$y_i \ge 0, \qquad i = 1, \dots, m$$

$$\min \; b^T y$$

$$s.t. \; A^T y \ge c$$

$$y \ge 0$$

The dual problem

To write the dual problem, have in mind the standard forms of minimization and maximization problems.

Primal Problem → **Dual Problem**

- ❑ Associate a dual variable to each constraint.
- ❑ The objective function coefficients of one problem are the RHSs of the complementary problem.
- ❑ The technical coefficients of each constraint are given by the technical coefficients of the associated variable.
- ❑ The signal of each variable in one problem is associated with the signal of one constraint in the other problem:
  - An equality constraint in one problem is associated with a free variable of the other problem.
  - If a constraint (resp. variable) has a correct signal in one problem, then the corresponding variable (resp. constraint) in the complementary problem also has the correct signal.

! "Correct signal" means that the signal is according to the standard form of the corresponding problem.

Primal Problem

$$\begin{aligned} \max \quad & 9x_1 + x_3 \\ s.t. \quad & x_1 + 5x_3 = 7 \quad \leftarrow y_1 \\ & 6x_2 \geq 1 \quad \leftarrow y_2 \ (✗) \\ & 4x_1 + 2x_2 + x_3 \leq 8 \quad \leftarrow y_3 \ (✓) \\ & x_1, x_2 \geq 0 \quad (✓) \\ & x_3 \leq 0 \quad (✗) \end{aligned}$$

Dual Problem

$$\begin{aligned} \min \quad & 7y_1 + y_2 + 8y_3 \\ s.t. \quad & y_1 + 4y_3 \geq 9 \quad (✓) \\ & 6y_2 + 2y_3 \geq 0 \quad (✓) \\ & 5y_1 + y_3 \leq 1 \quad (✗) \\ & y_1 \ free \\ & y_2 \leq 0 \quad (✗) \\ & y_3 \geq 0 \quad (✓) \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ s.t. \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \qquad \begin{aligned} \min \quad & b^T y \\ s.t. \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

Prop 1: The dual of the dual is the primal.

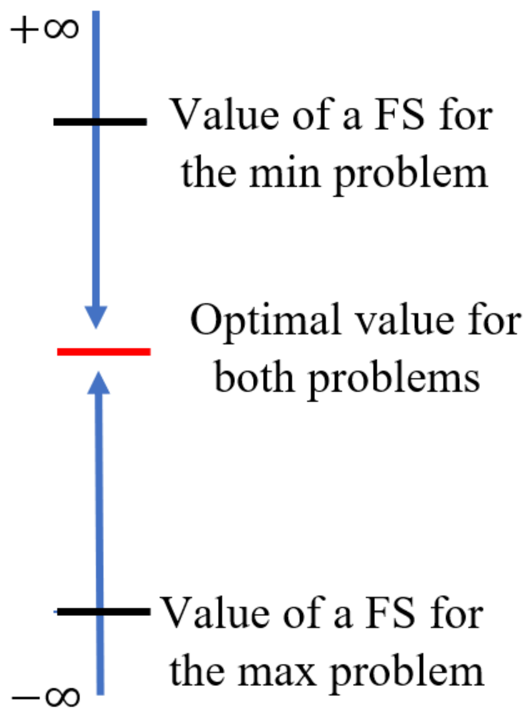
Prop 2: Given a pair of dual problems where

$\boldsymbol{x}$ is a feasible solution to the maximization problem

$\boldsymbol{y}$ is a feasible solution to the minimization problem

It holds Value($\boldsymbol{x}$) ≤ Value($\boldsymbol{y}$).

If Value($\boldsymbol{x}$) = Value($\boldsymbol{y}$), then $\boldsymbol{x}$ and $\boldsymbol{y}$ are optimal solutions for the corresponding problems.



Prop 3: If both problems have at least one feasible solution each, then both problems are bounded, and their optimal value coincides.

Prop 4: If one of the problems is unbounded, then the complementary problem is impossible.

Prop 5: If one of the problems is impossible, then the complementary problem is either impossible or unbounded.

Dual Solution (Shadow Prices)

Each dual variable y_i is associated with a specific constraint of the primal problem (constraint i) and its optimal value y_i^* is called **shadow price**.

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad \longleftarrow \quad (y_i)$$

The shadow price y_i^* represents the variation in the optimal value of the primal problem caused by increasing/decreasing the RHS of constraint i in one unit (if the optimal basis is kept).

The dual solution can be determined by solving the dual problem directly (graphical method, simplex method, or Excel Solver) or indirectly by solving the primal problem first:

i. If the primal problem was solved by the simplex method, see the optimal tableau.

ii. If the primal problem was solved by the solver, see the excel reports.

iii. If the primal problem was solved by the graphical method, see the graphic.

iv. If we have the primal solution, you can use the complementary slackness relations.

i. The dual solution can be read from the z-row of the optimal simplex tableau of the primal problem. It corresponds to the coefficients of the slack variables.

Primal Problem

$$\begin{aligned} \max \quad & z = 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

→ … →

Optimal simplex tableau

Basic Variables	z	x_1	x_2	s_1	s_2	s_3	RHS
z	1	0	0	0	3/2	1	36
s_1	0	0	0	1	1/3	-1/3	2
x_2	0	0	1	0	1/2	0	6
x_1	0	1	0	0	-1/3	1/3	2

↓

Dual Solution: $(y_1^*, y_2^*, y_3^*) = \left(0, \frac{3}{2}, 1\right)$

ii. The dual solution can be read from the shadow prices column displayed in the sensitivity report obtained when solving the primal problem by using the Excel solver.

Constraints

Name	Final Value	Shadow Price	Constraint Right side	Allowed Increase	Allowed Decrease
C1	2	0	4	1E+30	2
C2	12	1,5	12	6	6
C3	18	1	18	6	6

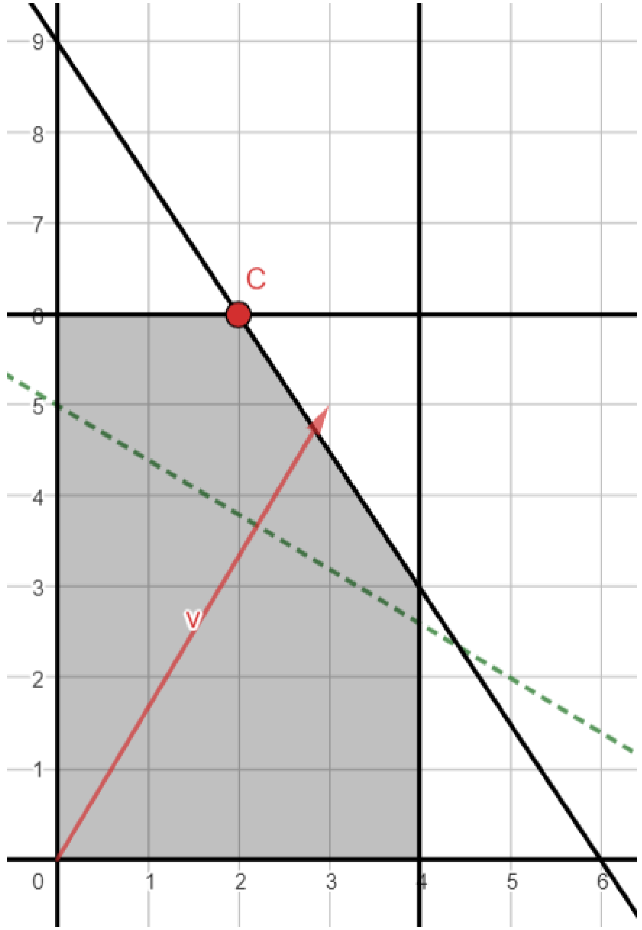
Dual Solution: $(y_1^*, y_2^*, y_3^*) = \left(0, \frac{3}{2}, 1\right)$

iii. Each shadow price is the variation in the optimal value caused by increasing the RHS of the corresponding primal constraint in one unit. By doing such an increase, we can determine the change in the optimal value to obtain the associated shadow price.

$$\begin{aligned} \max \quad & z = 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

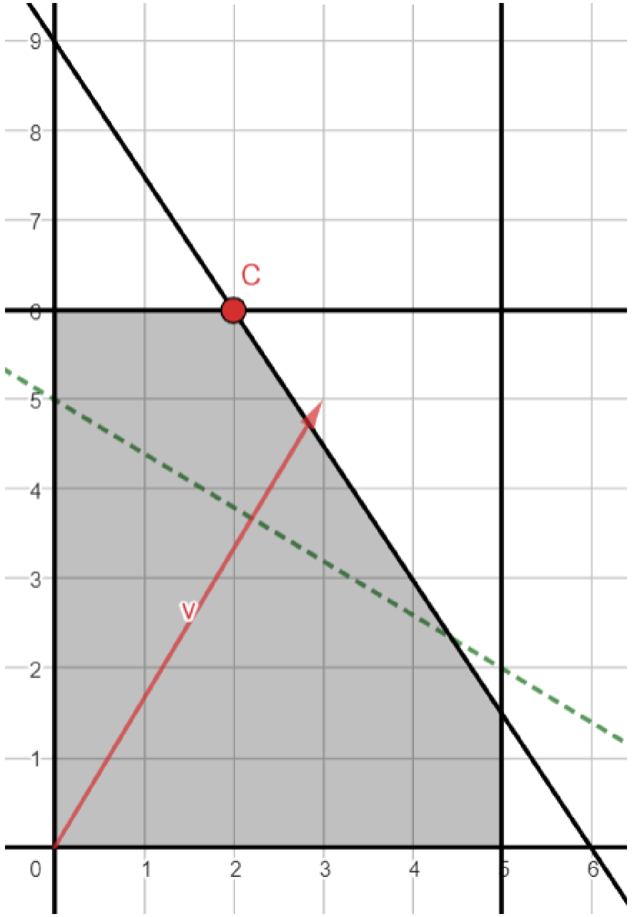
$$z^*_{old} = 36$$

$$y_i^* = z^*_{new} - z^*_{old} \qquad i = 1, 2, 3$$



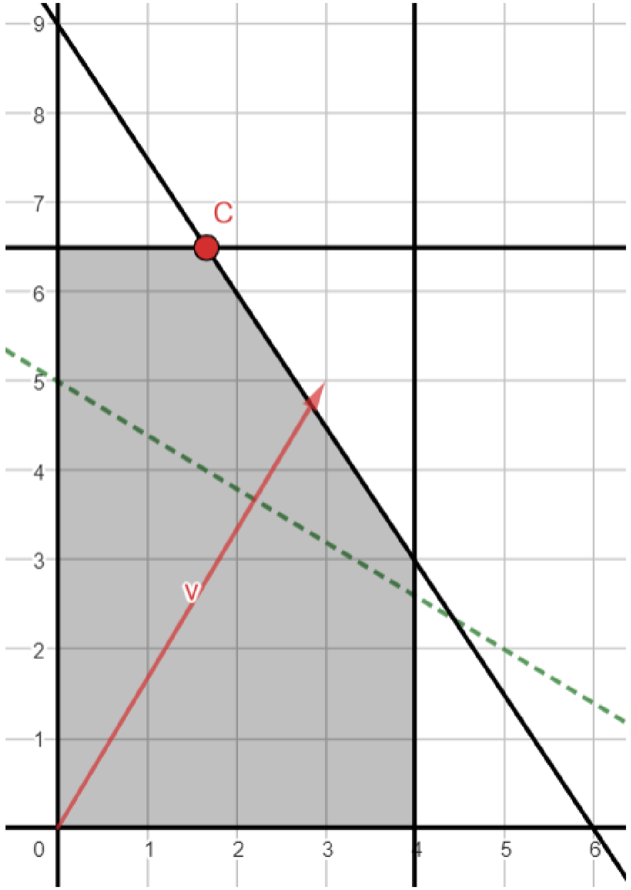
Determine the Dual Solution (iii)

$x_1 \leq \mathbf{5}$



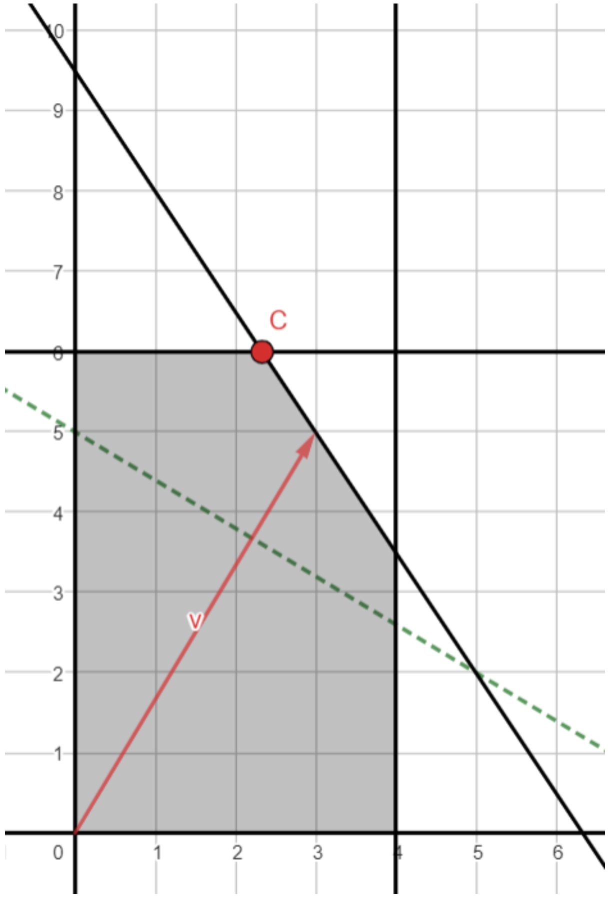
$y_1^* = 36 - 36 = 0$

$2x_2 \leq \mathbf{13}$



$y_2^* = 37.5 - 36 = 1.5$

$3x_1 + 2x_2 \leq \mathbf{19}$



$y_3^* = 37 - 36 = 1$

iv. Having the solution of the primal problem, we can use the **complementary slackness relations** to determine the dual solution.

Primal problem

$$\max \sum_{j=1}^{n} c_j x_j$$

$$s.t. \sum_{j=1}^{n} a_{ij}\, x_j \le b_i, \qquad i = 1, \dots, m \longleftarrow (y_i)$$

$$x_j \ge 0, \qquad j = 1, \dots, n$$

Dual problem

$$\min \sum_{i=1}^{m} b_i y_i$$

$$s.t. \sum_{i=1}^{m} a_{ji}\, y_i \ge c_j, \qquad j = 1, \dots, n \longleftarrow (x_j)$$

$$y_i \ge 0, \qquad i = 1, \dots, m$$

The complementary slackness relations are as follows:

$$y_i^* \left(b_i - \textstyle\sum_{j=1}^{n} a_{ij} x_j^*\right) = 0 \qquad \text{and} \qquad x_j^* \left(c_j - \textstyle\sum_{i=1}^{m} a_{ji} y_i^*\right) = 0$$

The Complementary Slackness relations are as follows:

$$y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j^* \right) = 0 \quad \text{and} \quad x_j^* \left(c_j - \sum_{i=1}^{m} a_{ji} y_i^* \right) = 0$$

$$\Leftrightarrow \quad y_i^* \times s_i = 0 \quad \text{and} \quad x_j^* \times u_j = 0$$

Hence:

- If a **primal constraint is not binding** in the optimal solution ($s_i \neq 0$), then the optimal value of the corresponding **dual variable is zero** ($y_i^* = 0$).

- If the optimal value of a **primal decision variable is not zero** ($x_j \neq 0$), then the corresponding **dual constraint is binding** $\left(c_j - \sum_{i=1}^{m} a_{ji} y_i^* = 0 \right)$.

For this example…

Primal

$$\begin{aligned} \max \quad & z = 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 \le 4 \quad \leftarrow y_1 \\ & 2x_2 \le 12 \quad \leftarrow y_2 \\ & 3x_1 + 2x_2 \le 18 \quad \leftarrow y_3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Dual

$$\begin{aligned} \min \quad & 4y_1 + 12y_2 + 18y_3 \\ \text{s.t.} \quad & y_1 + 3y_3 \ge 3 \quad \leftarrow x_1 \\ & 2y_2 + 2y_3 \ge 5 \quad \leftarrow x_2 \\ & y_1, y_2, y_3 \ge 0 \end{aligned}$$

The complementary slackness relations are:

$$\begin{cases} y_1^* \times s_1^* = 0 \ \text{ and } \ s_1^* \neq 0 \ \Rightarrow \ y_1^* = 0 \\ y_2^* \times s_2^* = 0 \\ y_3^* \times s_3^* = 0 \\ x_1^* \times u_1^* = 0 \ \text{ and } \ x_1^* \neq 0 \ \Rightarrow \ u_1^* = 0 \\ x_2^* \times u_2^* = 0 \ \text{ and } \ x_2^* \neq 0 \ \Rightarrow \ u_2^* = 0 \end{cases} \Rightarrow \begin{cases} y_1^* = 0 \\ ----- \\ ----- \\ y_1^* + 3y_3^* = 3 \\ 2y_2^* + 2y_3^* = 5 \end{cases} \Rightarrow \begin{cases} y_1^* = 0 \\ y_2^* = \frac{3}{2} \\ y_3^* = 1 \end{cases}$$

$$(x_1^*, x_2^*, s_1^*, s_2^*, s_3^*) = (2, 6, 2, 0, 0)$$

Original Problem

$$\begin{aligned} \max \quad & x_1 + 2x_2 + x_3 \\ s.t. \quad & x_1 + x_3 \le 10 \\ & 3x_1 - 2x_2 + x_3 \le 12 \\ & x_2 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution: $(6, 5, 4)$
Optimal value: $z^* = 20$

Modified Problem 1

$$\begin{aligned} \max \quad & x_1 + 2x_2 + x_3 \\ s.t. \quad & x_1 + x_3 \le 10 \\ & 3x_1 - 2x_2 + x_3 \le \mathbf{14} \\ & x_2 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

?

Modified Problem 2

$$\begin{aligned} \max \quad & x_1 + \mathbf{1}x_2 + x_3 \\ s.t. \quad & x_1 + x_3 \le 10 \\ & 3x_1 - 2x_2 + x_3 \le 12 \\ & x_2 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

?

Recall…

- A basic feasible solution of an LPP is composed of **basic variables** and **non-basic variables**.
- The set of basic variables is called the **basis** of the BFS.
- Two BFS are **adjacent** if their set of basic variables differs in exactly one variable.

A variation in a parameter of the original problem may change the optimal solution and/or the optimal value. If such a variation **keeps the basis** of the optimal solution, then we can determine the consequences of the change without solving the problem again.

The minimum and maximum values that a specific parameter can assume keeping the current basis optimal define the **sensitivity interval**.

Keep the basis means that the set of basic variables does not change (however, the value of the basic variables can change)

Sensitivity analysis for the RHS coefficients

Change:

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \quad \Rightarrow \quad \sum_{j=1}^{n} a_{ij}x_j \leq \boldsymbol{b_i^{new}}$$

In the sensitivity interval (SI) for parameter b_i:

- ✓ the shadow prices are kept
- ✓ the optimal solution may change
- ✓ the optimal value is changed according to the formula:

$$z_{new} = z_{old} + y_i^* \times \Delta_{b_i}$$

Sensitivity analysis for the O.F. coefficients

Change:

$$\min / \max \ldots\ c_j x_j \ \ldots \quad \Rightarrow \quad \min / \max \ldots\ \boldsymbol{c_j^{new}} x_j \ \ldots$$

In the sensitivity interval (SI) for parameter c_j:

- ✓ the optimal solution does not change
- ✓ the optimal value is changed according to the formula:

$$z_{new} = z_{old} + x_j^* \times \Delta_{c_j}$$

The sensitivity intervals in both cases can either be determined graphically or by using the solver reports.

The sensitivity intervals can be obtained from the sensitivity report provided by the Excel Solver

Variable Cells

Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
x1	2	0	3	4,5	3
x2	6	0	5	1E+30	3

$SI_{c_1} = [3-3\,,3+4.5] = [0\,,7.5]$

$SI_{c_2} = [5-3\,,5+\infty[\quad = [2\,,+\infty[$

Constraints

Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C1	2	0	4	1E+30	2
C2	12	1,5	12	6	6
C3	18	1	18	6	6

$SI_{b_1} = [4-2\,,4+\infty[\quad = [2\,,+\infty[$

$SI_{b_2} = [12-6\,,12+6] = [6\,,18]$

$SI_{b_3} = [18-6\,,18+6] = [12\,,24]$

To obtain the sensitivity interval for the RHS coefficient of constraint i

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

Step 1: Represent the feasible region of the problem.

Step 2: Move the line associated with constraint i in both directions while the optimal basis is kept. This may lead to a critical point from which the optimal basis changes or to the conclusion that the optimal basis never changes.

Step 2.1. In the former case, replace the critical point(s) found in the corresponding constraint(s):

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i^{min} \quad and/or \quad \sum_{j=1}^{n} a_{ij}x_j \leq b_i^{max}$$

to determine the value of b_i^{min} and/or b_i^{max}.

Step 2.2. In the later case, we have $b_i^{min} = -\infty$ or $b_i^{max} = +\infty$.

Step 3: The SI_{b_i} is defined by the values b_i^{min} and b_i^{max}.

Example:

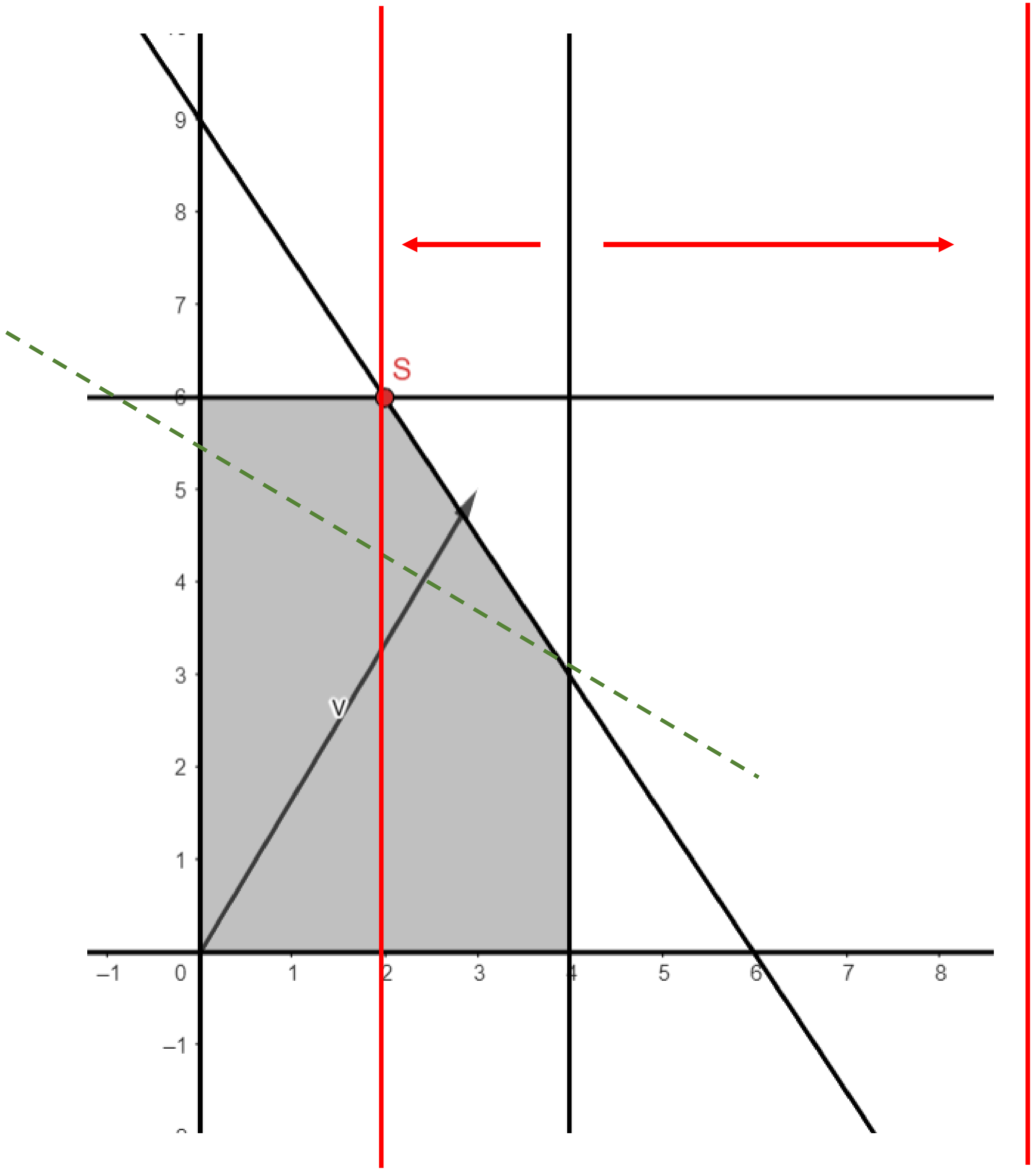
$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ s.t. \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

For the first RHS:

Critical point (on left) $(2,6) \rightarrow b_1^{min} = 2$

There is not a critical point on the right $\rightarrow b_1^{max} = +\infty$

$$\Rightarrow SI_{b_1} = [2, +\infty[$$



Example:

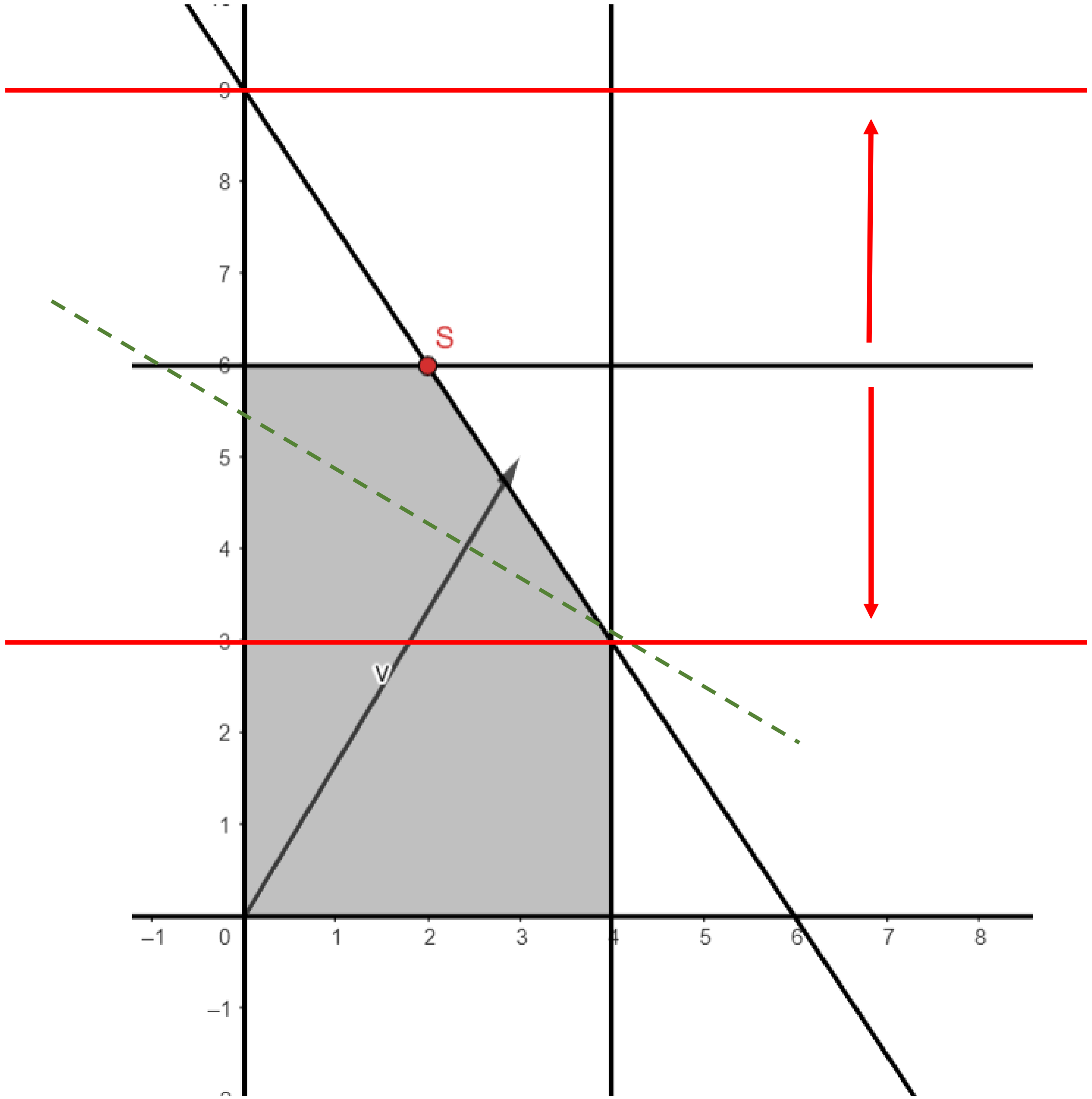
$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ s.t. \quad & x_1 \leq 4 \\ & 2x_2 \leq 12 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \end{aligned}$$

For the second RHS:

Critical point (on top) $(0,9) \rightarrow b_2^{max} = 18$

Critical point (on bottom) $(4,3) \rightarrow b_2^{min} = 6$

$$\Rightarrow SI_{b_2} = [6, 18]$$



Example:

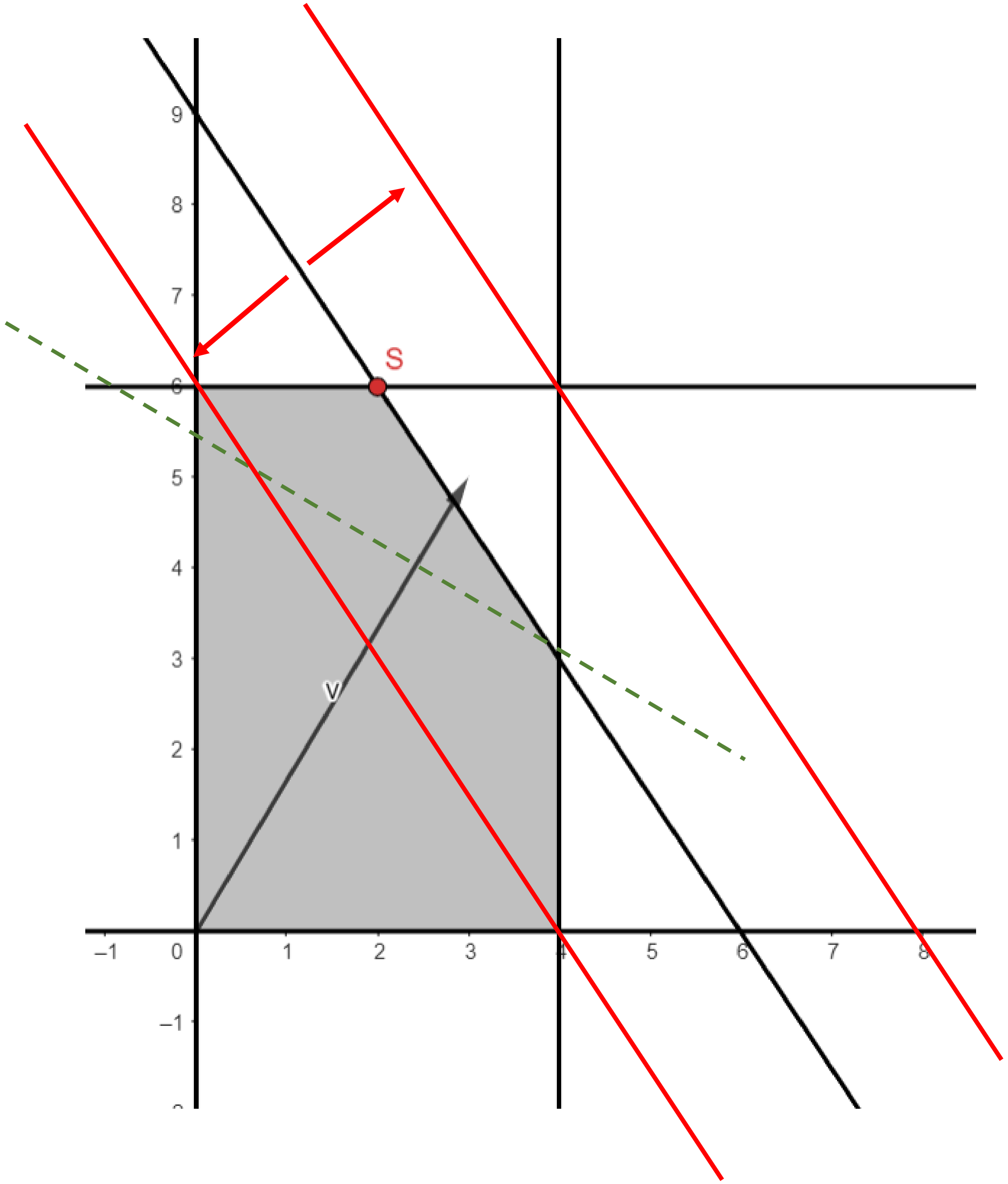
$$
\begin{aligned}
\max \quad & 3x_1 + 5x_2 \\
s.t. \quad & x_1 \leq 4 \\
& 2x_2 \leq 12 \\
& 3x_1 + 2x_2 \leq 18 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

For the third RHS:

Critical point (on top) $(4,6) \rightarrow \; b_3^{max} = 24$

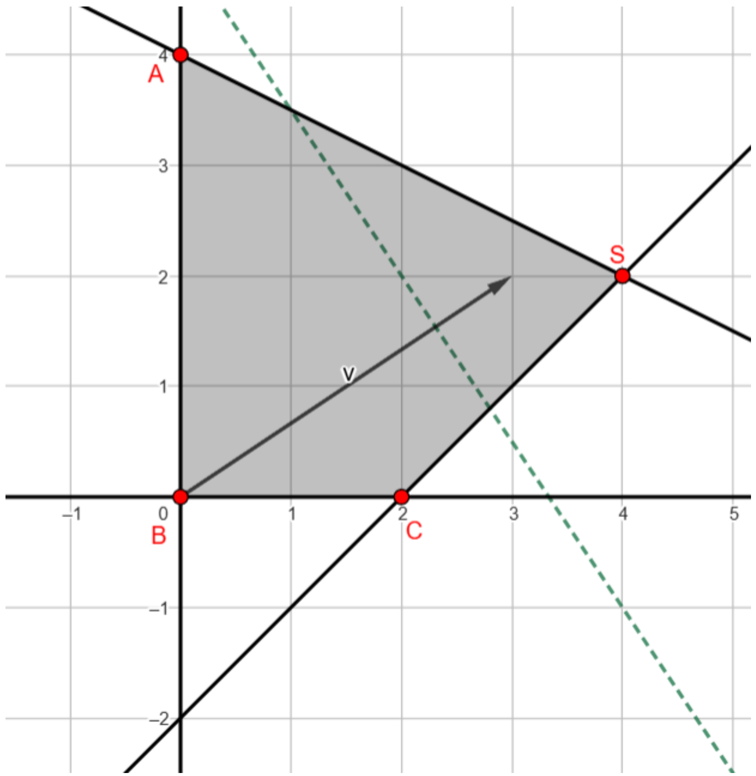
Critical point (on bottom) $(0,6) \rightarrow \; b_3^{min} = 12$

$$\Rightarrow \; SI_{b_3} = [12\,,24]$$



Example:

$$\begin{aligned} max \quad & 3x_1 + 2x_2 \\ s.t. \quad & x_1 + 2x_2 \leq 8 \\ & x_1 - x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{aligned}$$



Sensitivity interval for c_1

Consider max $c_1x_1 + 2x_2$

- **When c_1 decreases…**
  In a first moment, the slope of the green line becomes closer to the slope of $\overline{AS}$. At certain point, the slope of the green line equals the slope of $\overline{AS}$, making all points in that segment optimal. If c_1 continues to decrease, the slope of the green line becomes higher than the slope of segment $\overline{AS}$, and the optimal solution changes to point A. Thus,

$$slope_{O.F.} \leq slope_{\overline{AS}} \quad \Leftrightarrow \quad \frac{-c_1}{2} \leq \frac{-1}{2} \quad \Leftrightarrow \quad c_1 \geq 1$$

- **When c_1 increases…**
  The green line becomes more and more vertical, but the optimal solution is always point S regardless the increase. Thus, c_1 can increase as much as we want.

Thus: $SI_{c_1} = [1\,,+\infty]$

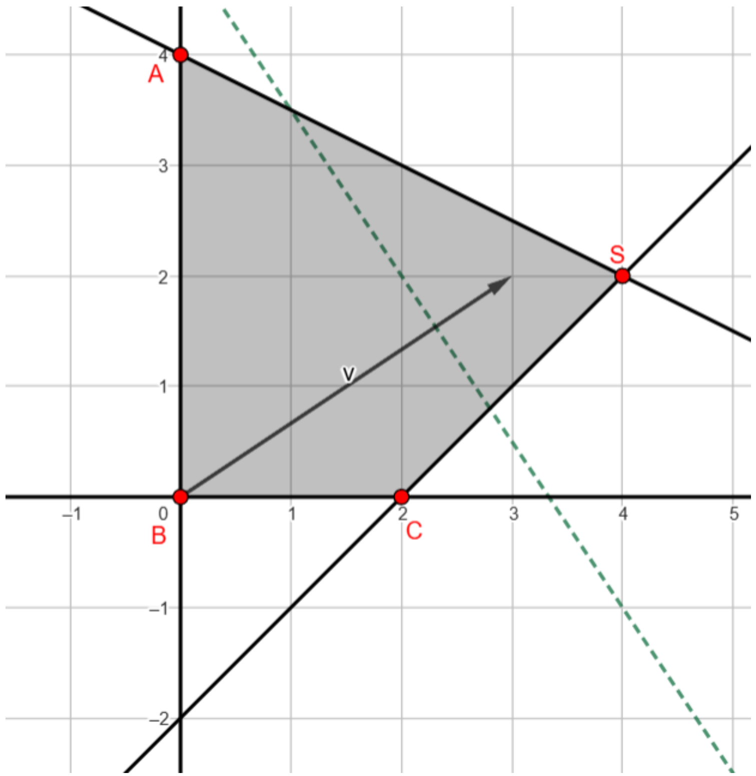
The slope of an affine function $ax_1 + bx_2 = c$ is $-a/b$.

Example: $\max \quad 3x_1 + 2x_2$

$s.t. \quad x_1 + 2x_2 \leq 8$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$



Sensitivity interval for c_2

Consider $\max 3x_1 + c_2x_2$

- **When c_2 increases…**
  The interpretation is similar to the first point in the previous slide. Thus,
  $$slope_{O.F.} \leq slope_{\overline{AS}} \iff \frac{-3}{c_2} \leq \frac{-1}{2} \iff c_2 \leq 6$$
- **When c_2 decreases…** but remains positive…
  The green line becomes more and more vertical, and the optimal solution is always point S. Thus, c_2 can decrease to zero.
- **When c_2 decreases…** and becomes negative…
  When c_2 is a very small negative value (-0.001, for example) the green line is almost vertical. If c_2 continues decreasing, point S remains optimal in a first moment. At certain point, the slope of the green line equals the slope of $\overline{SC}$, making all the points in that segment optimal. If c_2 continues to decrease, the optimal solution changes to point C. Thus,
  $$slope_{O.F.} \leq slope_{\overline{SC}} \iff \frac{-3}{c_2} \leq \frac{1}{1} \iff c_2 \geq -3$$

Thus: $SI_{c_2} = [-3\,, 6]$

Final Remarks:

We already know:

In the sensitivity interval (SI) for parameter b_i:

- ✓ the shadow prices are kept
- ✓ the optimal solution may change
- ✓ the optimal value is changed according to the formula:

$$z_{new} = z_{old} + y_i^* \times \Delta_{b_i}$$

In the sensitivity interval (SI) for parameter c_j:

- ✓ the optimal solution does not change
- ✓ the optimal value is changed according to the formula:

$$z_{new} = z_{old} + x_j^* \times \Delta_{c_j}$$

⚠ This means that we can only analyze the changes in the optimal value and/or optimal solution if the new value of the parameter suffering a change is within the sensitivity interval!

- Adding a new variable implies adding a new column to the original problem, which is equivalent to adding a new constraint to the dual.

$$
\begin{aligned}
\max \quad & c_1x_1 + \cdots c_nx_n + c^{new}x^{new} \\
s.t. \quad & a_{11}x_1 + \cdots a_{1n}x_n + a_1^{new}x^{new} \leq b_1 \\
& \vdots \\
& a_{m1}x_1 + \cdots a_{mn}x_n + a_m^{new}x^{new} \leq b_m \\
& x_1, \ \ldots, \ x_n, \ x^{mew} \geq 0
\end{aligned}
$$

$$\Rightarrow \quad a_1^{new}y_1 + \cdots + a_m^{new}y_m \geq c^{new}$$

- If the optimal solution of the original dual problem $(y_1^*, \ldots, y_m^*)$ satisfies this new constraint, then it does not change when introducing the new variable, and therefore, the introduction of the new variable is irrelevant.

LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

CHAPTER 4.

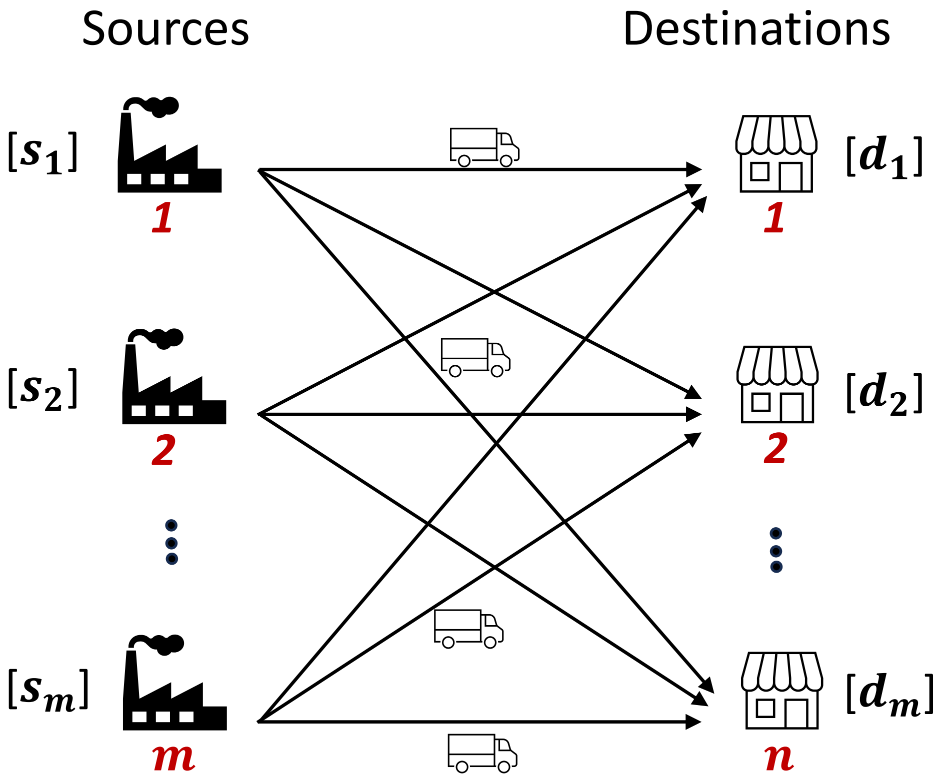
Transportation and Assignment Problems

Summary:

- Formulation of TPs/APs and their variants;
- Obtain feasible solutions for TPs/APs;
- Obtain the optimal solution of TPs/APs (by using the Solver);
- Properties and variants of TPs and APs.

*See H&L, Chapter 9, pages 318 – 327, 348 – 356

There is a product produced in m sources that must be sent to n destinations. Each connection between a source and a destination has a cost that depends on the quantity sent. The main goal is to determine the quantities to send from each source to each destination at the minimum cost.



Parameters:

s_i - supply at source i

d_j - demand of destination j

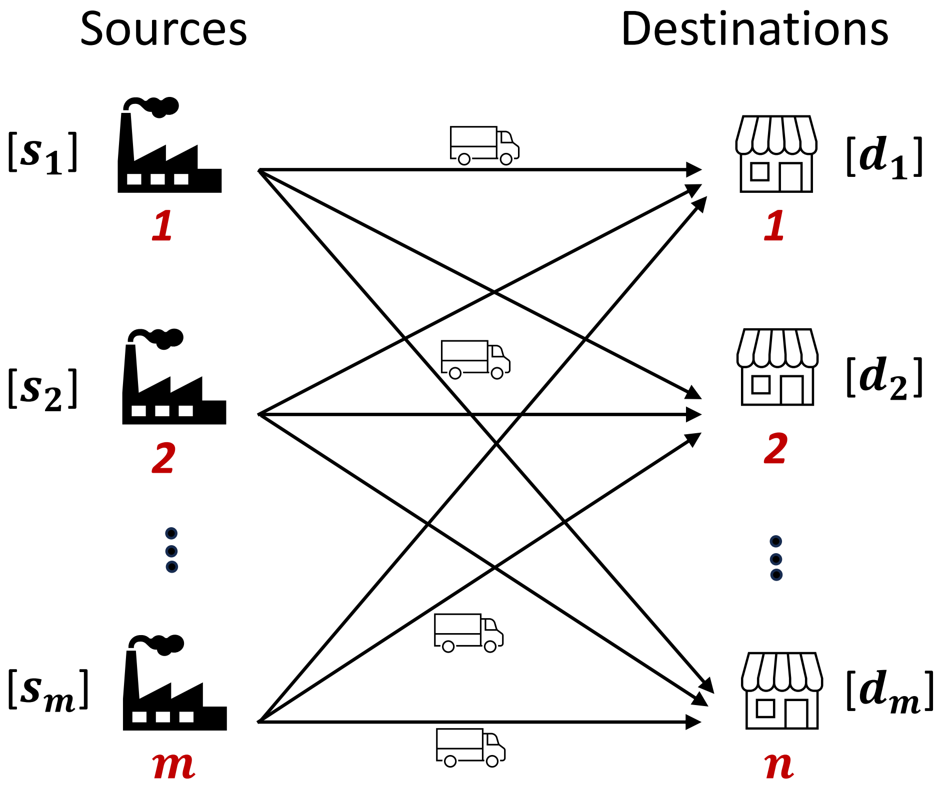
c_{ij} - cost of sending one unit of product from source i to destination j

Decision variables:

x_{ij} - quantity of product to send from source i to destination j.

If the total supply equals the total demand, the problem is **balanced** and:

- The total amount received by each destination is equal to its demand
- The total amount sent by each source is equal to its supply



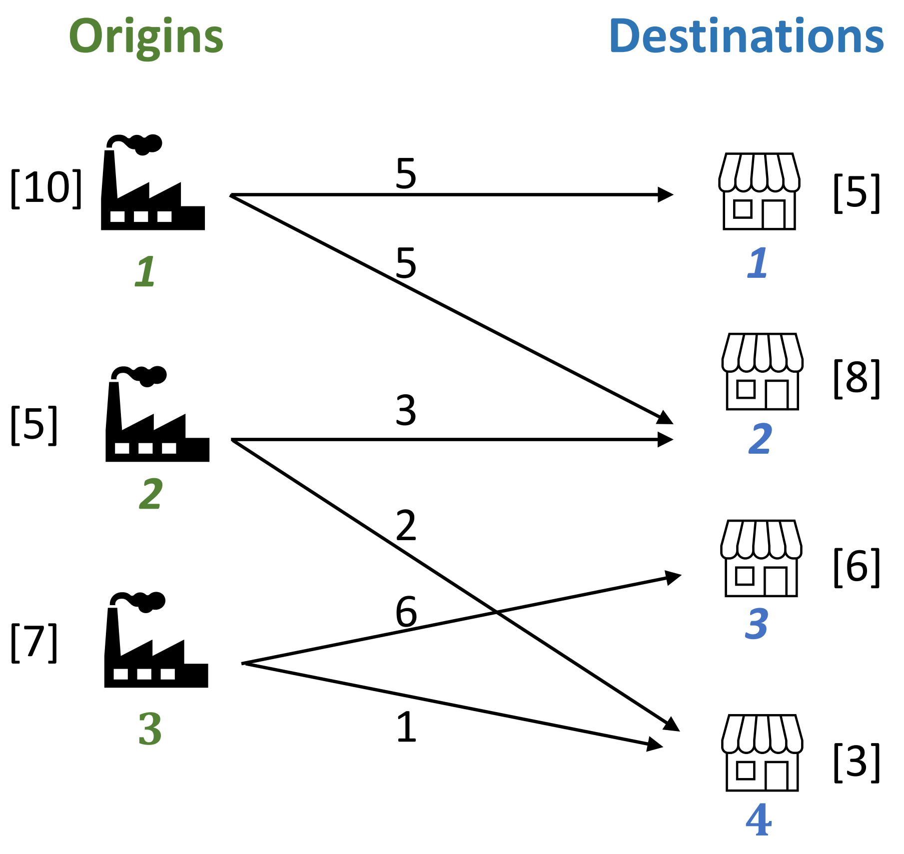
$$\min \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$s.t. \quad \sum_{i=1}^{m} x_{ij} = d_j, \quad j = 1, \dots, n$$

$$\sum_{j=1}^{n} x_{ij} = s_i, \quad i = 1, \dots, m$$

$$x_{ij} \geq 0, \quad i = 1, \dots, m, \; j = 1, \dots, n$$

Feasible solution



Data:

Costs	D1	D2	D3	D4	Supply
O1	2	5	0	7	**10**
O2	1	6	3	4	**5**
O3	5	8	1	9	**7**
Demand	**5**	**8**	**6**	**3**	

Example of a feasible solution:

	D1	D2	D3	D4	Supply
O1	5	5			**10**
O2		3		2	**5**
O3			6	1	**7**
Demand	**5**	**8**	**6**	**3**	

Total Cost = 76

Optimal solution (by using the Excel Solver)

	A	B	C	D	E	F	G	H	I
1									
2		**Costs**	D1	D2	D3	D4			
3		O1	2	5	0	7			
4		O2	1	6	3	4			
5		O3	5	8	1	9			
6									
7		**Solution**	D1	D2	D3	D4			**Supply**
8		O1					0	=	10
9		O2					0	=	5
10		O3					0	=	7
11			0	0	0	0			
12			=	=	=	=			
13		**Demand**	5	8	6	3			
14									
15		**O.F.**	0						

=SUM(C8:F8)
=SUM(C9:F9)
=SUM(C10:F10)

=SUMPRODUCT(C3:F5;C8:F10)

=SUM(C8:C10) =SUM(D8:D10) =SUM(E8:E10) =SUM(F8:F10)

Optimal solution (by using the Excel Solver)

	A	B	C	D	E	F	G	H	I
1									
2		**Costs**	D1	D2	D3	D4			
3		O1	2	5	0	7			
4		O2	1	6	3	4			
5		O3	5	8	1	9			
6									
7		**Solution**	D1	D2	D3	D4			**Supply**
8		O1	3	7			10	=	10
9		O2	2			3	5	=	5
10		O3		1	6		7	=	7
11			5	8	6	3			
12			=	=	=	=			
13		**Demand**	5	8	6	3			
14									
15		**O.F.**	69						
16									
17									
18									
19									
20									
21									
22									
23									
24									

Solver Parameters

Set Objective: $C$15

To: ○ Max ◉ Min ○ Value Of: 0

By Changing Variable Cells: $C$8:$F$10

Subject to the Constraints:

$C$11:$F$11 = $C$13:$F$13
$G$8:$G$10 = $I$8:$I$10

Add | Change | Delete | Reset All | Load/Save

☑ Make Unconstrained Variables Non-Negative

Select a Solving Method: Simplex LP | Options

Solving Method

Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.

Help | Solve | Close

TP Variants and Properties

Variants of TPs:

- Total supply > Total demand $\Longrightarrow$ " $\leq$ " constraints in the sources
- Total supply < Total demand $\Longrightarrow$ " $\leq$ " constraints in the destinations
- Minimum and maximum demands $\Longrightarrow$ A " $\leq$ " constraint and a " $\geq$ " constraint for each destination
- Minimum and maximum supplies $\Longrightarrow$ A " $\leq$ " constraint and a " $\geq$ " constraint for each source
- Impossible links between source i and destination j $\Longrightarrow$ Impose $x_{ij} = 0$ or define $c_{ij} = \infty$

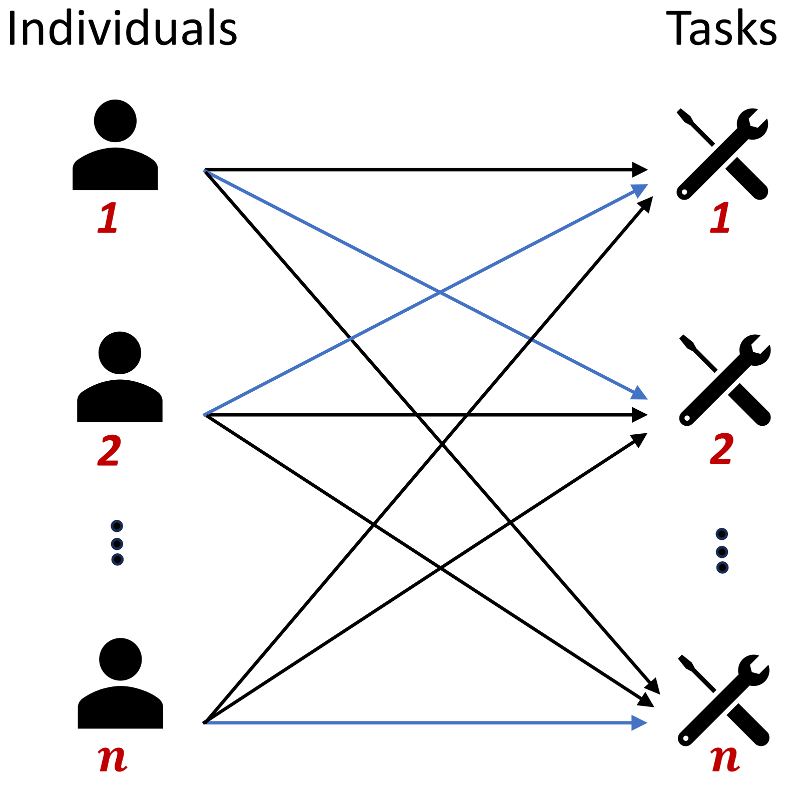
Properties of TPs

Prop 1: The TP has at least one feasible solution, and consequently it has an optimal solution.

Prop2: A TP where all supplies and demands are integer values has at least one integer optimal solution, that is, a solution where all variables assume integer values.

The Assignment Problem (AP)

The problem consists of assigning $\boldsymbol{n}$ individuals to $\boldsymbol{n}$ tasks to minimize the total cost of assignment.



Parameters:

c_{ij} - cost of assigning person i to task j

Decision variables:

$$x_{ij} = \begin{cases} 1 & \text{if person } i \text{ is assigned to task } j \\ 0 & \text{otherwise} \end{cases}$$

LP Formulation

$$\min \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$s.t. \quad \sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, \dots, n$$

$$\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, \dots, n$$

$$x_{ij} \in \{0,1\}, \quad i, j = 1, \dots, n$$

Remarks:

- Due to the structure of the problem, the binary constraints can be replaced by non-negativity constraints

$$x_{ij} \in \{0, 1\}, \qquad i, j = 1, \dots, n \quad \Longrightarrow \quad x_{ij} \geq 0, \qquad i, j = 1, \dots, n$$

- The AP is a particular case of the TP

Variants of AP:

- \# of individuals > # of tasks $\Longrightarrow$ " $\leq$ " constraints for the individuals
- \# of individuals < # of tasks $\Longrightarrow$ " $\leq$ " constraints for the tasks
- Impossible assignment of individual i to task j $\Longrightarrow$ Impose $x_{ij} = 0$ or define $c_{ij} = \infty$

CHAPTER 5.

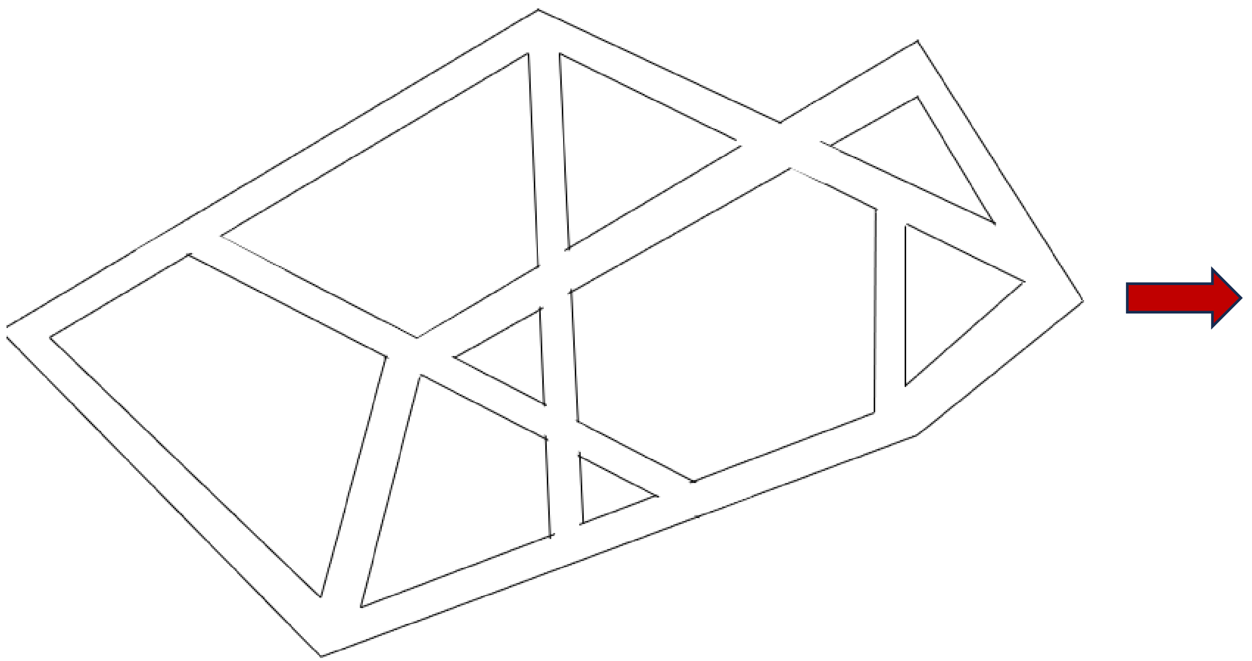
Network Optimization Problems

Summary:

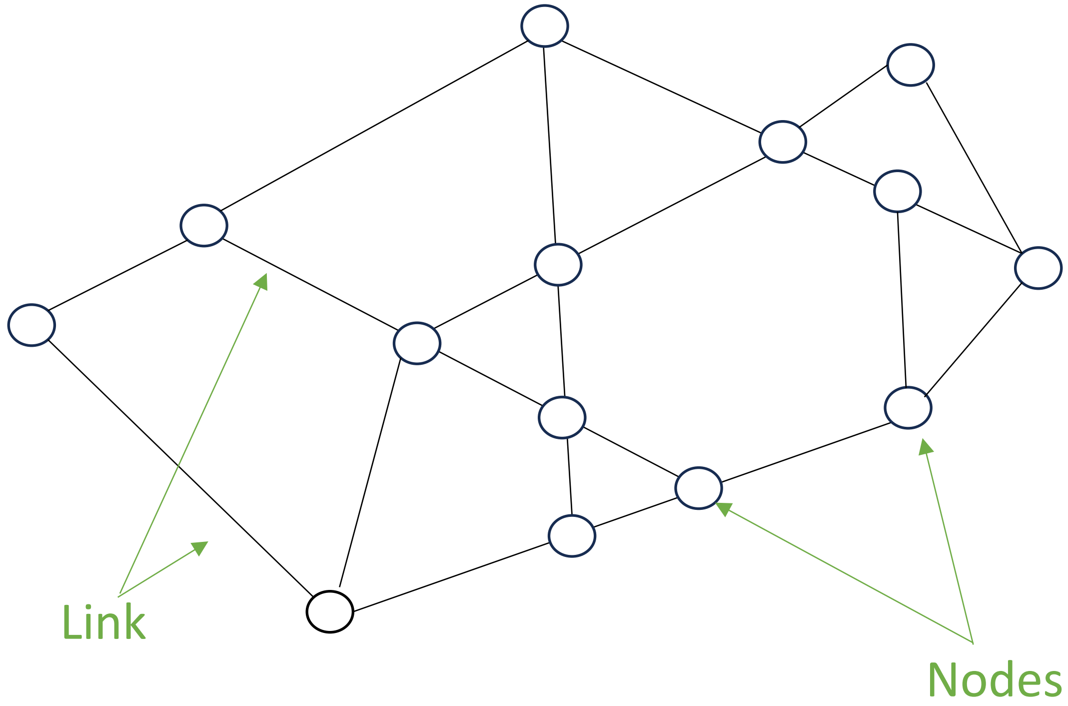
- Definitions associated with networks;
- Minimum Cost Flow problem;
- Shortest Path problem;
- Minimum Spanning Tree problem;
- Feasible solutions, optimal solutions, properties, and relations between the problems.

*See H&L, Chapter 10 pages 372 – 436

A **network** (or graph) is an ordered pair $G = (V, A)$ where V is the set of nodes or vertices and A is a set of links connecting pairs of nodes.

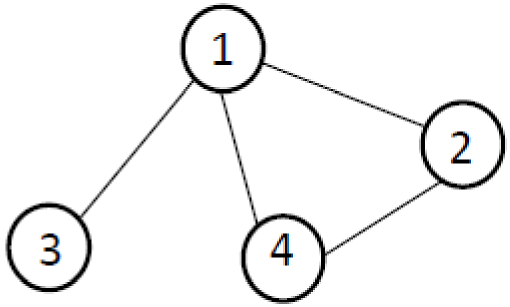


Road Map



Network

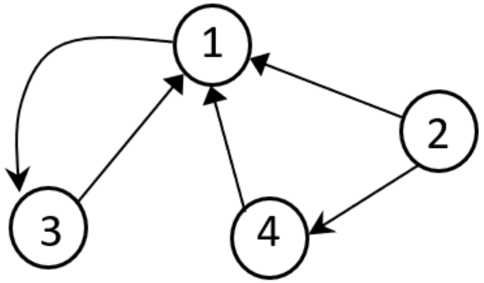
Undirected network



The links are called **edges** and are represented by (i, j) or $\{i, j\}$, where i and j are adjacent nodes, also called extremities.

A **path** between i and j is a sequence of distinct edges connecting these nodes.

Directed network



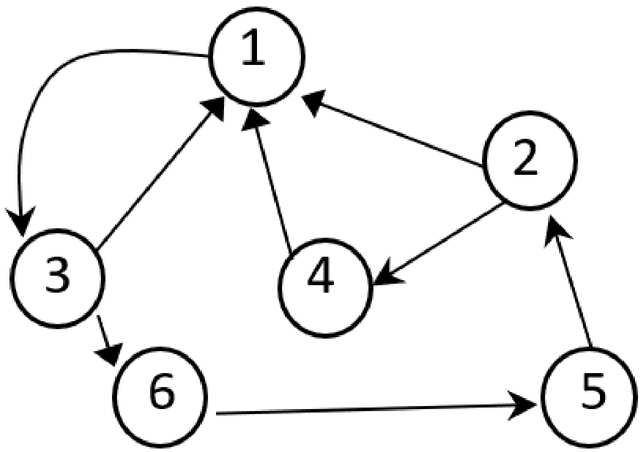
The links are called **arcs** and are represented by (i, j) or $i \rightarrow j$, where i is the predecessor of j and j is the successor of i.

A **directed path** from i and j is a sequence of distinct edges connecting these nodes, toward j.

A network with both directed and undirected links is called **mixed network**.

For both Directed and Undirected networks:

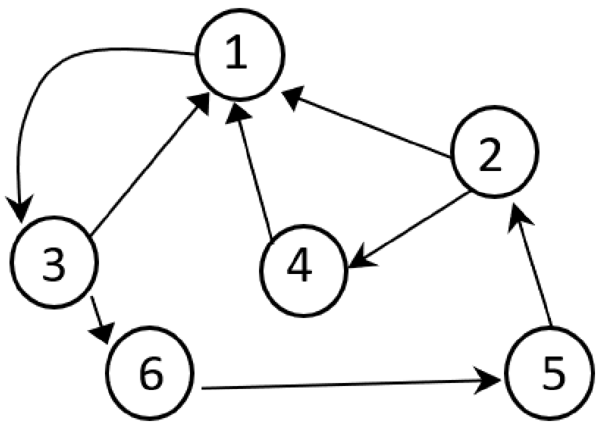
- Edge/Arc (i, j) is **incident** in nodes i and j.
- An **undirected path** from node i to node j is a sequence of connecting arcs/edges whose direction (if any) can be either toward or away from node j.
- A **cycle** is a path that begins and ends in the same node.



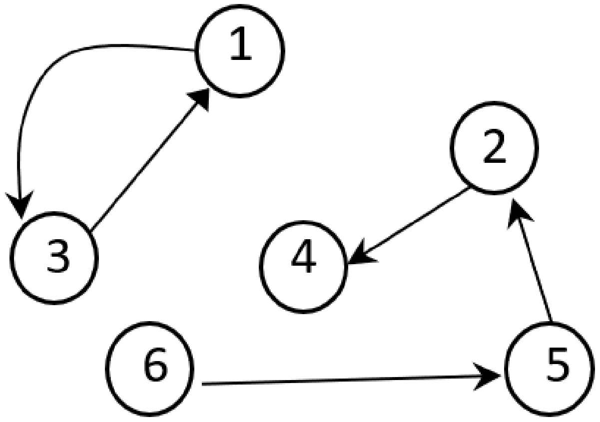
$\big((1,3),(3,6),(6,5),(5,2),(2,1)\big)$ → Directed path (and cycle)

$\big((3,6),(3,1),(1,2),(2,4)\big)$ → Undirected path between 6 and 4

- Two nodes are **connected** if the network contains at least one undirected path between them.
- A network is **connected** if every pair of nodes is connected.



Connected network



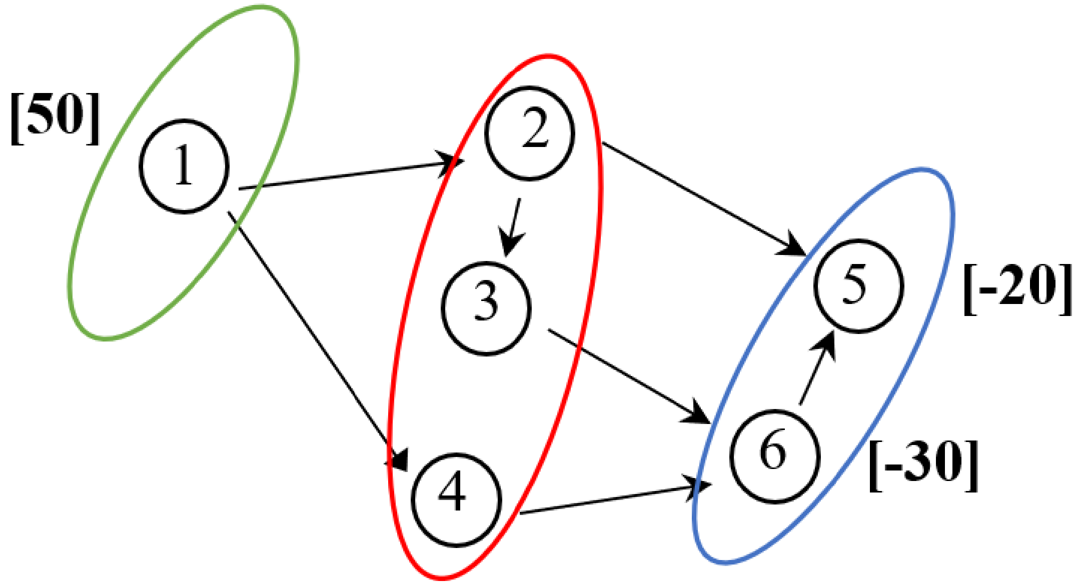
Network not connected

- The **flow** in a directed network is the amount of "product" that crosses its arcs.

Supply / Source nodes → Flow generator node (Outflow > Inflow)

Transshipment / Intermediate nodes → Flow conservation node (Outflow = Inflow)

Demand / Destination / Sink nodes → Flow consumer node (Outflow < Inflow)



Let $G = (V, A)$ be a directed and connected network with at least one supply node and at least one destination node, being the remaining nodes transshipment nodes.

The **MCFP** consists of determining how to send the available supply from the supply nodes to the destination nodes to satisfy the demand at the minimum cost (c_{ij}) by respecting arc capacities (u_{ij}),

Parameters:

c_{ij} - cost of sending one flow unit through arc (i, j)

u_{ij} - Maximum flow quantity in arc (i, j)

b_i - Flow generated by node i

Decision variables:

x_{ij} : Flow that crosses arc (i, j)

LP Formulation

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij}$$

$$s.t. \quad \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = b_i \qquad i \in V$$

$$0 \le x_{ij} \le u_{ij} \qquad (i,j) \in A$$

Flow out – Flow in = Generated flow

Assumptions of the MCFP

- The network is directed and connected.
- Arc capacities are compatible with supplies and demands.
- In a **balanced** MCFP, the total supply is equal to the total demand ($\sum_{i \in V} b_i = 0$).

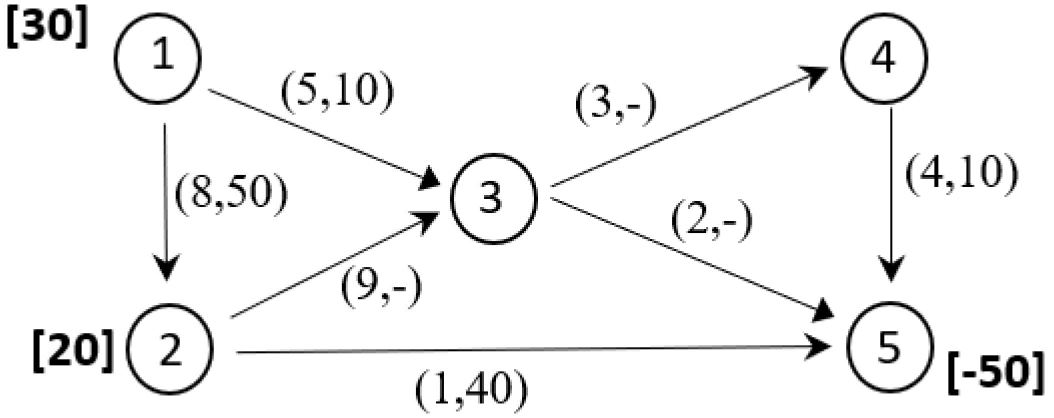
Properties of the MCFP

- The MCFP has at least one feasible solution, and therefore, it also has an optimal solution.
- A MCFP where all b_i and u_{ij} are integer values has, at least, one integer optimal solution.

MCFP Variants

- Total supply > Total demand $\Longrightarrow$ " $\leq$ " constraints in the sources
- Total supply < Total demand $\Longrightarrow$ " $\leq$ " constraints in the destinations
- Maximization problem

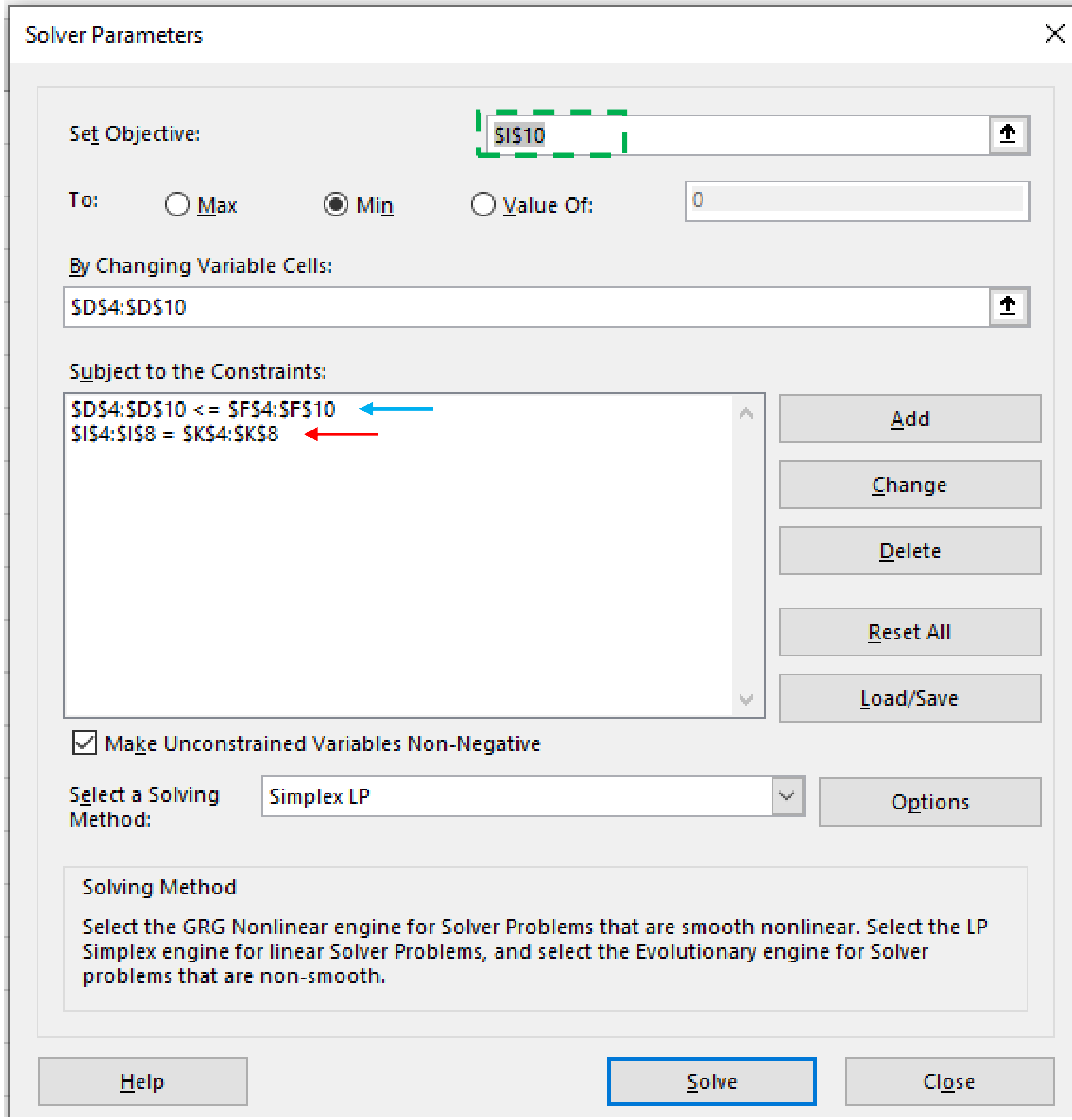
TPs are particular cases of MCFPs!



Constraints
=SUMIF($B$4:$B$10;H4;$D$4:$D$10) - SUMIF($C$4:$C$10;H4;$D$4:$D$10)
=SUMIF($B$4:$B$10;H5;$D$4:$D$10) - SUMIF($C$4:$C$10;H5;$D$4:$D$10)
=SUMIF($B$4:$B$10;H6;$D$4:$D$10) - SUMIF($C$4:$C$10;H6;$D$4:$D$10)
=SUMIF($B$4:$B$10;H7;$D$4:$D$10) - SUMIF($C$4:$C$10;H7;$D$4:$D$10)
=SUMIF($B$4:$B$10;H8;$D$4:$D$10) - SUMIF($C$4:$C$10;H8;$D$4:$D$10)
=SUMPRODUCT(D4:D10;E4:E10)

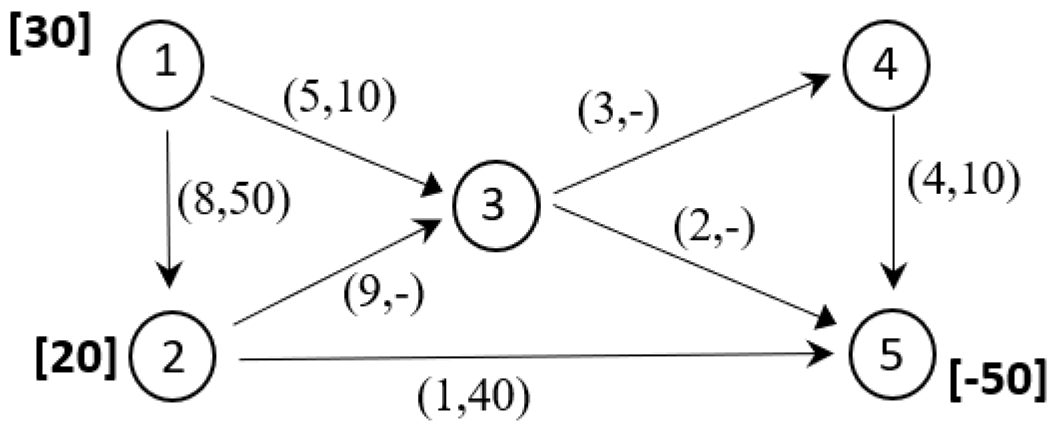
	A	B	C	D	E	F	G	H		J	K	L
1												
2											Generated	
3		From	To	Flow	Cost	Capacity		Node	Constraints	Signal	Flow	
4		1	2		8	50		1	0	=	30	
5		1	3		5	10		2	0	=	20	
6		2	3		9	1000		3	0	=	0	
7		2	5		1	40		4	0	=	0	
8		3	4		3	1000		5	0	=	-50	
9		3	5		2	1000						
10		4	5		4	10		O.F	0			

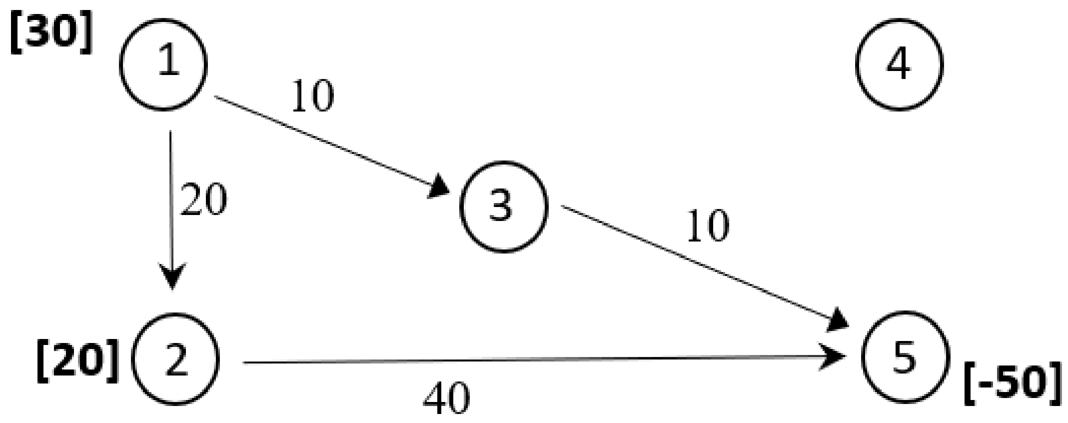
From	To	Flow	Cost	Capacity		Node	Constraints	Signal	Generated Flow
1	2		8	50		1	0	=	30
1	3		5	10		2	0	=	20
2	3		9	1000		3	0	=	0
2	5		1	40		4	0	=	0
3	4		3	1000		5	0	=	-50
3	5		2	1000					
4	5		4	10		O.F	0		



Part I – Minimum Cost Flow Problem (MCFP)

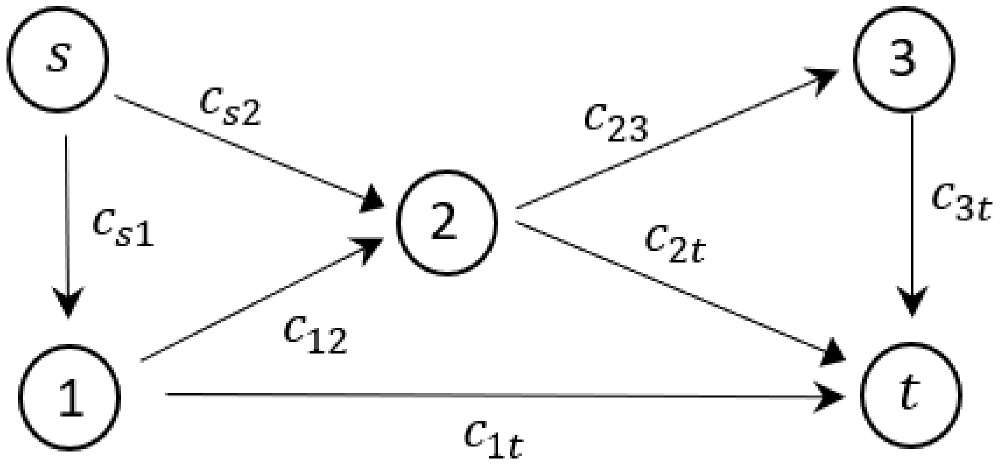
	A	B	C	D	E	F	G	H	I	J	K	L
1												
2											Generated	
3		From	To	Flow	Cost	Capacity		Node	Constraints	Signal	Flow	
4		1	2	20	8	50		1	30	=	30	
5		1	3	10	5	10		2	20	=	20	
6		2	3		9	1000		3	0	=	0	
7		2	5	40	1	40		4	0	=	0	
8		3	4		3	1000		5	-50	=	-50	
9		3	5	10	2	1000						
10		4	5		4	10		O.F	270			
11												

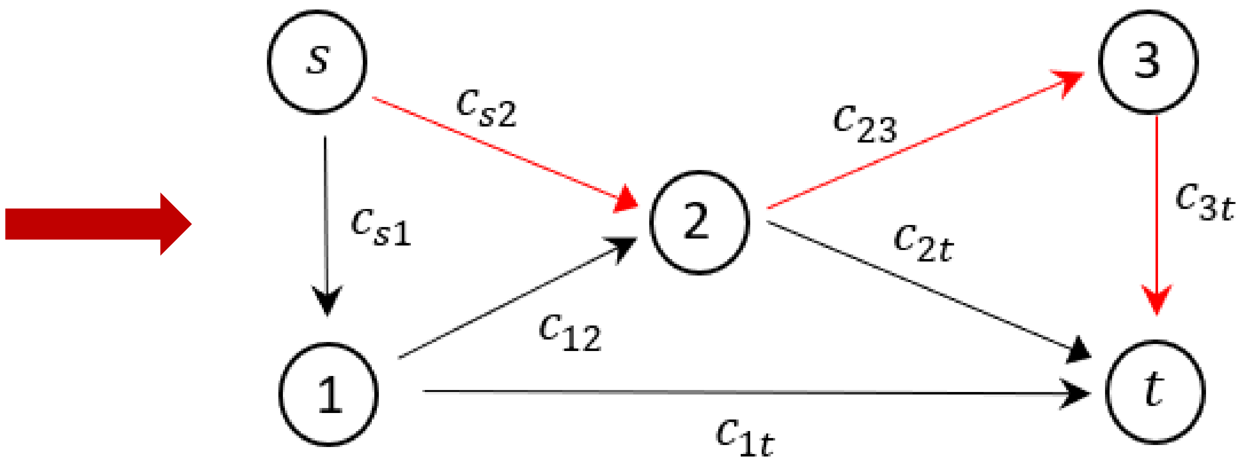




Let G=(V, A) be a directed and connected network with only one origin (s) and one destination (t).

The **Shortest Path Problem (SPP)** consists of determining the path with the minimum distance (c_{ij}) between such an origin and such a destination.





The SPP is a particular case of the MCFP where there is only one source, one destination, and one unit of flow to send and no capacities. Therefore, the idea behind the formulation is similar.

Parameters:

c_{ij} - cost associated with arc (i,j)

Decision variables:

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is in the path} \\ 0 & \text{otherwise} \end{cases}$$

LP Formulation

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij}$$

$$s.t. \quad \sum_{j:(s,j)\in A} x_{sj} = 1 \quad \text{(node } s \text{)}$$

$$-\sum_{j:(j,t)\in A} x_{jt} = -1 \quad \text{(node } t \text{)}$$

$$\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0, \quad i \in V\backslash\{s,t\} \quad \text{(others)}$$

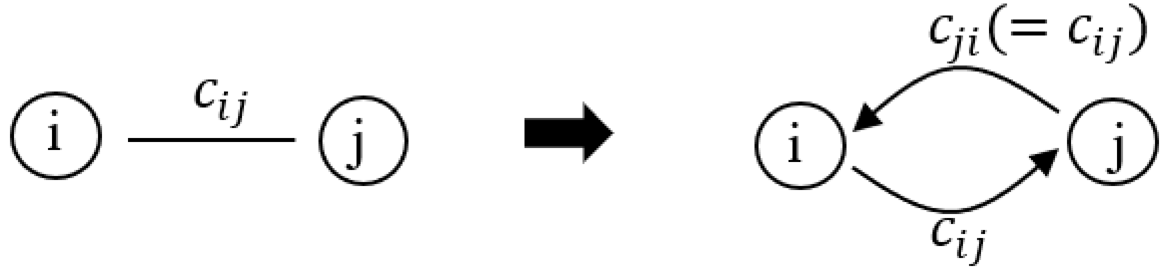
$$x_{ij} \in \{0,1\} \qquad (i,j) \in A$$

Remarks

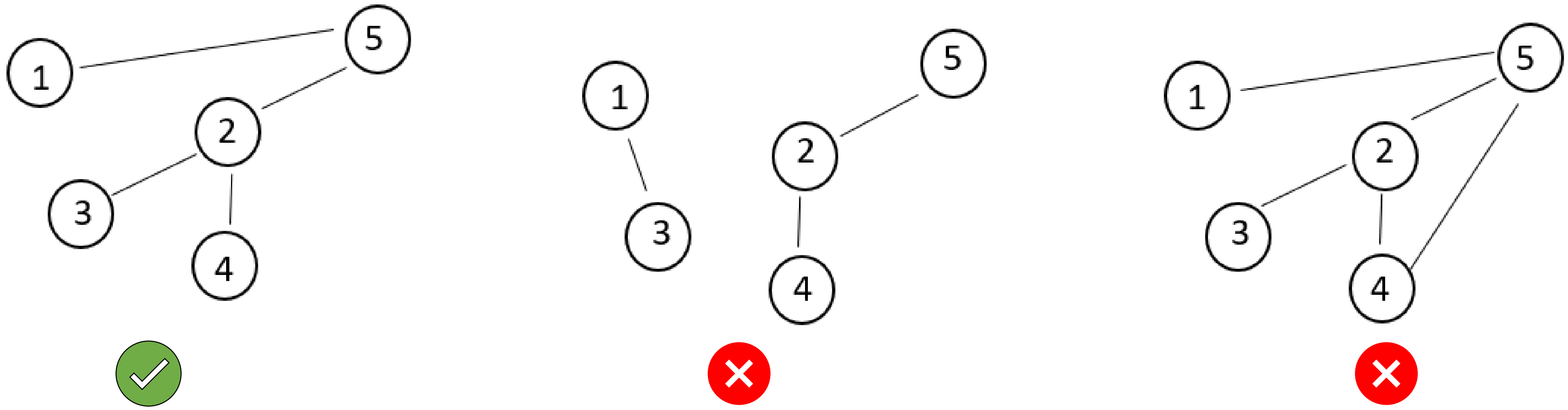
- In the LP model, if each constraint $x_{ij} \in \{0,1\}$ is replaced by $0 \leq x_{ij} \leq 1$, then at least one integer optimal solution exists.

- If $c_{ij} > 0$ for all $(i,j) \in A$, then the constraints $x_{ij} \in \{0,1\}$ can be replaced by constraints $x_{ij} \geq 0$.

- Any undirected link can be converted into two directed links



Let $G = (V, A)$ be an undirected and connected network with lengths (c_{ij}) associated to the edges.
A **spanning tree** of network G is a connected network containing all nodes V and without cycles.



The **Minimum Spanning Tree Problem (MSTP)** consists of choosing the set of edges that represents the spanning tree having the minimum total length

- A spanning tree of a network with *n* nodes has the same *n* nodes and *n*-1 edges.
- The MST can be determined by the Prim Algorithm.

Prim Algorithm

Step 1. Choose any node. Initialize the tree with that node.

Step 2. **If** all nodes are in the tree, **then**

Go to Step 3.

Else

Select the shortest edge linking a node outside the tree to a node already in the tree.

Add the edge to the tree.

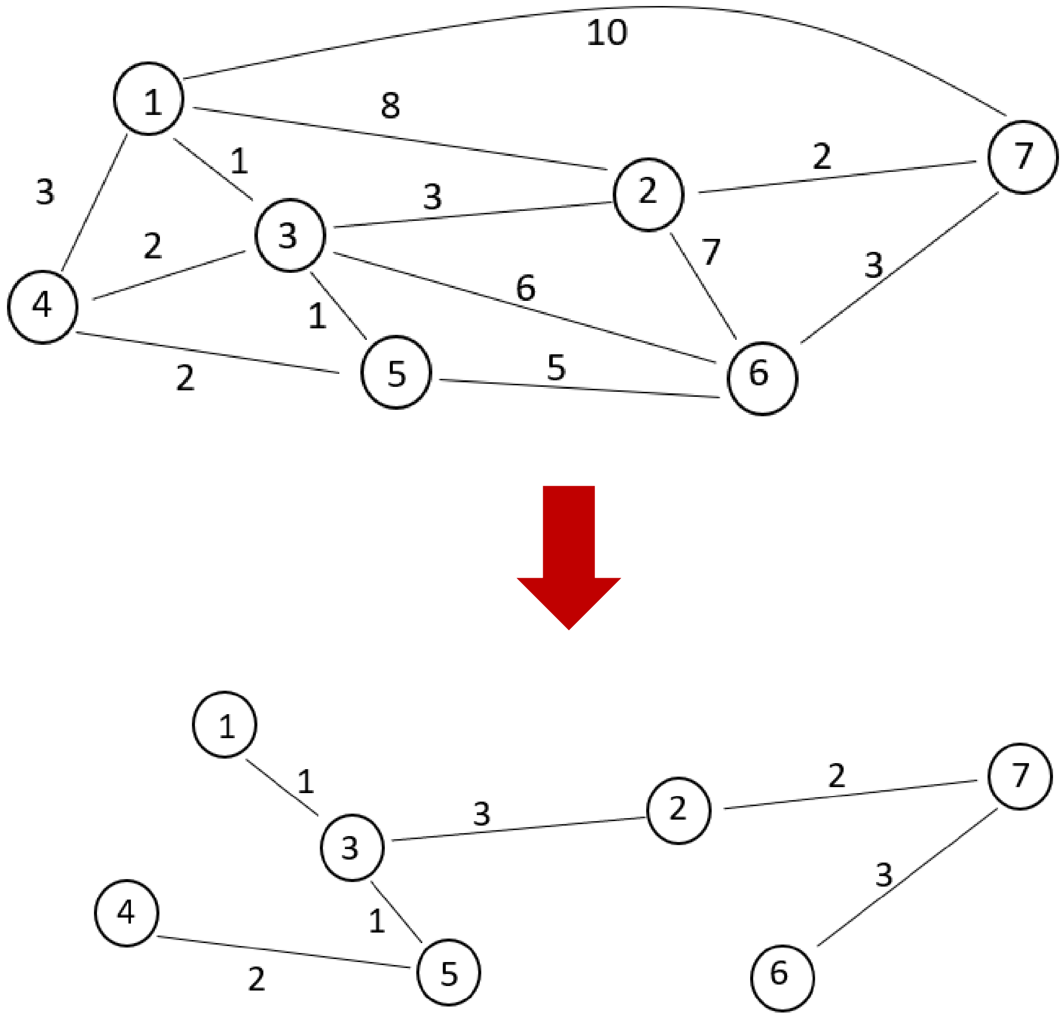
Go to Step 2.

Step 3. Draw the minimum spanning tree and determine its total length.

Part III – Minimum Spanning Tree Problem (MSTP)

Iteration	Nodes in the tree	Adjacent closest node not in the tree	Edge length	Edge to include in the tree
1	4*	5	2	(4,5)
2	4	3	2	(5,3)
	5	3	1	
3	4	1	3	(1,3)
	5	6	5	
	3	1	1	
4	4	-	-	(3,2)
	5	6	5	
	3	2	3	
	1	2	8	
5	4	-	-	(2,7)
	5	6	5	
	3	6	6	
	1	7	10	
	2	7	2	
6	4	-	-	(6,7)
	5	6	5	
	3	6	6	
	1	-	-	
	2	6	7	
	7	6	3	

*arbitrary choice.



LISBON SCHOOL OF ECONOMICS & MANAGEMENT
UNIVERSIDADE DE LISBOA

CHAPTER 6.

Integer Linear Programming

Summary:

- Integer Linear Programming problems;
- LP-relaxation of an ILP problem;
- LP formulations with binary variables.

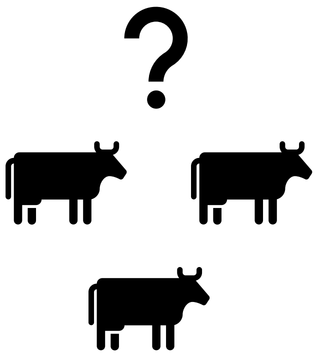
*See H&L, Chapter 12 pages 474 – 501

Integer Linear Programming Problem

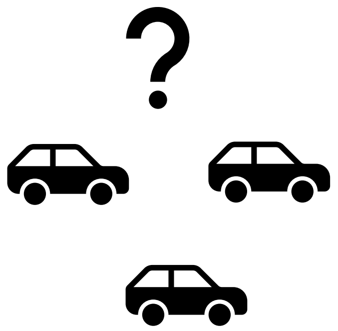
An **Integer Linear Programming** (ILP) problem is an optimization problem containing decision variables that can only assume integer values.

Examples:

y – "Number of cows to raise"



x – "Number of cars to buy"



An ILP problem can be classified as:

- **Pure ILP** problem: when all decision variables are integer
- **Mixed ILP** problem: when just a subset of decision variables are integer

LP - relaxation

ILP problem

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad & i = 1, \dots, m \\ & x_j \in \mathbb{Z}_0^+, \quad & j = 1, \dots, n \end{aligned}$$

Only the integer points in the region defined by the constraints are feasible

Optimal value: z_{ILP}^*

is not better than

Linear Programming Relaxation (LP-relaxation)

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad & i = 1, \dots, m \\ & x_j \ge 0, \quad & j = 1, \dots, n \end{aligned}$$

All points in the region defined by the constraints are feasible

Optimal value: z_R^*

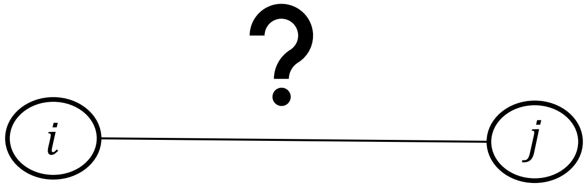
Binary Integer Programming (BIP) problems are ILP problems containing binary decision variables.

A BIP problem can be classified as:

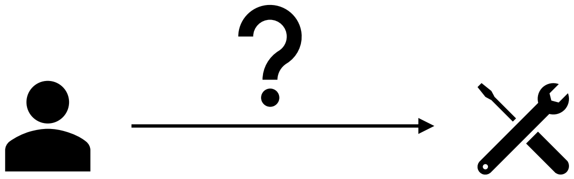
- **Pure BIP** problem: when all decision variables are binary
- **Mixed BIP** problem: when just a subset of decision variables are binary

Examples: Shortest Path problems and Assignment problems

$$x_{ij} = \begin{cases} 1 & \text{if arc } (i,j) \text{ is in the path} \\ 0 & \text{otherwise} \end{cases}$$



$$x_{ij} = \begin{cases} 1 & \text{if person } i \text{ is assigned to task } j \\ 0 & \text{otherwise} \end{cases}$$



1. **Mutually exclusive products:**

Let x_1 and x_2 be the quantities of products P_1 and P_2 to produce, and assume that at most one of the products can be produced.

Option 1

Define a new binary variable:

$$y_i = \begin{cases} 1 & \text{if } P_i \text{ is produced} \\ 0 & \text{otherwise} \end{cases} \qquad i = 1,2$$

and impose the constraints:

$$y_1 + y_2 \leq 1$$
$$x_i \leq M y_i \quad i = 1,2$$
$$y_i \in \{0,1\} \quad i = 1,2$$

where M is a sufficient large value.

Option 2

Define a new binary variable:

$$z = \begin{cases} 1 & \text{if } P_1 \text{ is produced (and } P_2 \text{ is not produced)} \\ 0 & \text{if } P_2 \text{ is produced (and } P_1 \text{ is not produced)} \end{cases}$$

and impose the constraints:

$$x_2 \leq M(1 - z)$$
$$x_1 \leq Mz$$
$$z \in \{0,1\}$$

where M is a sufficient large value.

2. Alternative constraints:

Consider that only one of the following constraints must be satisfied:

$$\text{(C1)}\ \ LHS_1 \le RHS_1 \qquad \text{or} \qquad \text{(C2)}\ \ LHS_2 \le RHS_2.$$

Define a new binary variable:

$$y = \begin{cases} 1 & \text{if constraint (C1) is satisfied (active)} \\ 0 & \text{if constraint (C2) is satisfied (active)} \end{cases}$$

and impose the constraints:

$$LHS_1 \le RHS_1 + M(1-y)$$

$$LHS_2 \le RHS_2 + My$$

$$y \in \{0,1\}$$

where M is a sufficient large value.

3. Setup costs:

Let us assume that the quantity to produce of product P is given by variable x. A setup cost (s) is a fixed cost that must be paid to produce the product regardless of the quantity to produce .

Define a new binary variable:

$$y = \begin{cases} 1 & \text{if product } P \text{ is produced} \\ 0 & \text{otherwise} \end{cases}$$

and impose the constraints:

$$\max \quad z - s \times y$$
$$s.t \quad x \leq My$$
$$y \in \{0,1\}$$

where M is a sufficient large value and z is the original objective function.

In a minimization problem the setup cost is added to the objective function

1. **Other situations:**

Consider the production of two products (P_1 and P_2) and the following binary variables:

$$y_i = \begin{cases} 1 & \text{if product } P_i \text{ is produced} \\ 0 & \text{otherwise} \end{cases} \qquad i = 1,2$$

Then:

- Only one of the products is produced → $y_1 + y_2 = 1$
- At most one of the products is produced → $y_1 + y_2 \leq 1$
- Either both products are produced or none of them is produced → $y_1 = y_2$
- Producing P_1 implies to produce P_2 → $y_1 \leq y_2$
  (Or: if P_2 is not produced then P_1 cannot be produced as well)

Many other situations can be modeled with binary variables…