1. a) Main variables: $x_M^* = 2; x_F^* = 2; x_C^* = 0$; Slack variables: $s_1^* = 12; s_2^* = s_3^* = 0$. Optimal value: $Z^* = 8$. To obtain the maximum utility (=8), 2 tons of medicines and 2 tons of food should be shipped ($x_M^* = x_F^* = 2$). Clothes should not be shipped ($x_F^* = 0$). There are still 12 units of capacity of the selection team that are not used ($s_1^* = 12$). All the capacity of the packaging team is used ($s_2^* = 0$). The tons of medicines to send are exactly the minimum amount required ($s_3^* = 0$).

1. b) Dual

$$Min\ W = 20y_1 + 10y_2 + 2y_3$$

$$\text{s.a:} \begin{cases} 3y_1 + 3\,y_2 + y_3 \geq 2 \\ y_1 + 2y_2 \geq 2 \\ y_1 + y_2 \geq 1 \\ y_1, y_2 \geq 0\,, y_3 \leq 0 \end{cases}$$

1. c) Complementary slackness relations:

$$\begin{cases} x_M^* \times (2 - 3y_1^* - 3y_2^* - y_3^*) = 0 \\ x_F^* \times (2 - y_1^* - 2y_2^*) = 0 \\ x_C^* \times (1 - y_1^* - y_2^*) = 0 \\ y_1^* \times (20 - 3x_M^* - x_F^* - x_C^*) = 0 \\ y_2^* \times (10 - 3x_M^* - 2x_F^* - x_C^*) = 0 \\ y_3^* \times (2 - x_M^*) = 0 \end{cases} \Rightarrow \ \dots \ \Leftrightarrow \begin{cases} 2 - 3y_1^* - 3y_2^* - y_3^* = 0 \\ 2 - y_1^* - 2y_2^* = 0 \\ y_1^* = 0 \end{cases} \ \dots \ \Leftrightarrow \begin{cases} y_2^* = 1 \\ y_3^* = -1 \\ y_1^* = 0 \end{cases}$$

The slack variables can be determined from the dual problem considering the optimal dual solution. We have: $y_4^* = 0, y_5^* = 0, y_6^* = 0$.

1. d) $y_1 = 0$ means that if the optimal basis is kept, increasing or decreasing the capacity of the selection team does not change the total utility.

1. e) The RHS of the third constraint changes from 2 to 0 ($b_3 = 0$).

$0 \in SI_{b_3} = [0; 3{,}33]$ $\Longrightarrow \Delta Z = y_3^* \times \Delta b_3 = -1 \times (0 - 2) = 2$. The total utility increases 2 units.

1. f) The o.f coefficient of the first variable changes from 2 to 1 ($c_1 = 1$).

$1 \in SI_{c_1} =]-\infty; 3]$ $\Longrightarrow \Delta Z = x_M^* \times \Delta c_M = 2 \times (1 - 2) = -2$. The optimal solution is kept and the total utility decreases 2 units. The new optimal value is 6.

2.a) For example, assign V1 to T1, V2 to T2 and V3 to T3. Total time: $1 + 1 + 4 = 6$.

2.b) The value of any feasible solution is an upper bound for the optimal value. So, from item a) we can say that 6 is an upper bound.

2.c) Let us define: $x_{ij} = \begin{cases} 1 \; if \; voluntear \; V_i \; is \; assign \; to \; task \; T_j \\ 0 \qquad\qquad\qquad otherwise \end{cases}, i = 1,2,3,4, j = 1,2,3.$

$$Min\ Z = x_{11} + 3x_{12} + 4x_{13} + 2x_{21} + x_{22} + 3x_{23} + x_{13} + 4x_{32} + 4x_{33} + 3x_{42} + 2x_{43}$$

$$\text{s.t:} \begin{cases} x_{11} + x_{12} + x_{13} \leq 1 & \mathbf{V1} \\ x_{21} + x_{22} + x_{23} \leq 1 & \mathbf{V2} \\ x_{31} + x_{32} + x_{33} \leq 1 & \mathbf{V3} \\ x_{42} + x_{43} = 1 & \mathbf{V4} \\ x_{11} + x_{21} + x_{31} = 1 & \mathbf{T1} \\ x_{12} + x_{22} + x_{32} + x_{42} = 1 & \mathbf{T2} \\ x_{13} + x_{23} + x_{33} + x_{43} = 1 & \mathbf{T3} \\ x_{22} \leq x_{31} & \\ x_{ij} \in \{0,1\} \; i = 1,2,3,4; j = 1,2,3 & \end{cases}$$

3.a) Minimum Spanning Tree problem.

3.b) The set of links is a spanning tree because all nodes are connected and there are no cycles, but it is not the minimum spanning tree because it is possible to obtain a different spanning tree with a lower cost. For example, if we remove the link (A,F), which has a distance of 10, and add the link (B,F), which has a distance of 1, we find a the new spanning tree with a lower total distance.

3.c) Prim algorithm:

Iteration	Node in the tree	Adjacent closest node ∉ tree	link	minimum	Link in the tree
1	A	E	2	2	(A,E)
2	A	B	3		
	E	D	2	2	(E,D)
3	A	B	3	3	(A,B)
	E	F	5		
	D	B	4		
4	A	C	5		
	E	F	5		
	D	C	5		
	B	F	1	1	(B,F)
5	A	C	5		
	E	C	12		
	D	C	5		
	B	C	3	3	(B,C)
	F	C	18		

Optimal solution: $\{(A, E), (E, D), (A, B), (B, F), (B, C)\}$, total time =11 h.

4. (Theoretical question) See the notes of the first class about simplex method.