Mathematics I → 13/05/2026

Regular Assessment - Version A

Part I

a) $\operatorname{int} S =]2,3[$

$\left\{ 1-\frac{1}{2^n},\ n \in \mathbb{N} \right\}$

$\inf S = \frac{1}{2}$

S is not closed

b) $r = \frac{1}{2}$ $\quad v_1 = -16$

monotonic increasing

$\lim v_n = 0$

$$\sum_{n=3}^{+\infty} v_n = \frac{v_3}{1-r} = \frac{-4}{1-\frac{1}{2}} = \frac{-4}{1/2} = -8$$

c) $\operatorname{Im} f = [-4, 2]$

zeros of $f \circ g$: e^{-1} and e.

$$\lim_{n \to +\infty} f\left(-\frac{1}{n}\right) = 2.$$

②

d) $f'(x) + 2x\, f(x)^3 + x^2 \cdot 3 f(x)^2 \cdot f'(x) = 0$

$x = 1$ $\quad f'(1) + 2 f(1)^3 + 1^2 \cdot 3 f(1)^2 f'(1) = 0$

$\Longrightarrow f'(1) + 2 \cdot 2^3 + 3 \cdot 2^2 f'(1) = 0$

$\Longleftrightarrow 13 f'(1) = -16 \Leftrightarrow \boxed{f'(1) = \frac{-16}{13}}$.

e) $\dfrac{f(4) - f(1)}{4-1} = f'(x)$, $x \in]1,4[$

$f(4) \geqslant f'(x) \cdot 3 + f(1)$

$f(4) \geqslant 2 \cdot 3 + 5$

$\boxed{f(4) \geqslant 11}$

f) $f'(x) = \underline{-\sin x} \cdot e^{\cos x - 1}$

$f'(0) = 0$

$f''(x) = -\cos x\, e^{\cos x - 1} \underbrace{- \sin x \cdot (e^{\cos x - 1})'}_{=0}$.

$f''(0) = -1$

$$\boxed{P_2(x) = 1 - \frac{1}{2}\cdot x^2}$$

- $f(0,2) \approx 1 - \frac{1}{2}\cdot 0.2^2 = 0,98$

- $|f(0,2) - P_2(0.2)| \leq \frac{f'''(x)}{3!}\cdot(0.2)^3 \leq \frac{0.197}{6} \times 0.2^3$

g) $F(\sqrt{5}) = 25$

$F'(x) = 4x^3 - \left(\int_5^{x^2} e^{-t^2}\,dt\right)$

$= 4x^3 - 2x\cdot\left[e^{-x^4}\right]$

$F''(x) = 12x^2 - 2e^{-x^4} - 2x\left(e^{-x^4}\right)'$

$F''(0) = -2 < 0$

F has a relative maximum.

h) • $\bar{u} + (-1, 8, -8)$

• $k = 0$

• $\|\vec{w}\| = \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

det is 6

• $\bar{u}, \bar{v}$ and $\bar{w}$ are linearly independent

i) $\det A^T = -5$

$\det(3A) = 3^3 \cdot (-5)$

$\det \begin{bmatrix} a & b & c \\ 2d-3a & 2e-3b & 2f-3c \\ g & h & i \end{bmatrix} = 2\cdot(-5) = -10$

j) $r(A|B) = 2$

system is undefined with degree of freedom 1.

Part II

1) $n = 1$ $\quad \sum_{i=1}^{1} i^2 = 1 = \frac{1\cdot(1+1)(2+1)}{6}$

$1 = 1$ ✓

(4)

Hypothesis: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Thesis $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

Proof. $\sum_{i=1}^{n+1} i^2 \underset{\substack{\uparrow \\ \text{Hyp.}}}{=} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$

(6)

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)\left(n(2n+1)+6(n+1)\right)}{6} = \frac{(n+1)\left(2n^2+7n+6\right)}{6}$$

$$= \frac{(n+1)(2n+3)(n+2)}{6}.$$ ✓

2a)
(2) • f is continuous at $x \in \mathbb{R}^+$ because $f(x) = \frac{e^x - 1}{x}$ is the ratio of continuous maps whose denominator does not vanish

(2) • f is continuous at $x \in \mathbb{R}^-$ because it is the sum of continuous maps.

(6) • f is continuous at $x = 0$ because

• $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \frac{e^x - 1}{x} = \lim\limits_{x \to 0} \frac{e^x}{1} = 1$

• $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \cos x + \beta x^2 = 1$

• $f(0) = 1$

$$\lim_{x \to 0} f(x) = f(0) = 1.$$

b)
$$f'_+(0) = \lim_{x\to 0} \frac{\frac{e^x-1}{x}-1}{x} = \lim_{x\to 0} \frac{e^x-1-x}{x^2} = \frac{0}{0}$$

$$\underset{\substack{\uparrow\\ \text{L'Hôpital}}}{=} \lim_{x\to 0} \frac{e^x-1}{2x} \underset{\substack{\uparrow\\ \text{L'Hôpital}}}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

$$f'_-(0) = \lim_{x\to 0} \frac{\cos x + \beta x^2 - 1}{x} = \frac{0}{0}$$

$$\underset{\substack{\uparrow\\ \text{L'Hôpital}}}{=} \lim_{x\to 0} \frac{\sin x + 2\beta x}{1} = 0.$$

Since $f'_+(0) \neq f'_-(0)$, then f is not differentiable at $x=0$, $\forall \beta \in \mathbb{R}$.

c) f is continuous at $x \in [-5,5]$ (a)

$[-5,5]$ is compact

$\Downarrow$ Weierstrass theorem

$f\big|_{[-5,5]}$ has an absolute maximum

3 a)

$$g(x) = \frac{x^2+x+4}{x(x^2+4)}$$

$$\lim_{x\to 0^-} g(x) = -\infty$$

$$\lim_{x\to 0^+} g(x) = +\infty \quad (3)$$

$$\lim_{x\to \pm\infty} g(x) = 0 \quad \text{(both sides)} \quad (3)$$

Vertical Asymptote: $x=0$
horizontal asymptote $y=0$ } (4)

b) $$g(x) = \frac{x^2+x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

Solving system: $A = C = 1$. (8) and $B=0$.

$$\int g(x)\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{x^2+4}\,dx =$$

$$= \ln|x| + \int \frac{1/4}{\left(\frac{x}{2}\right)^2+1}\,dx$$

$$= \ln|x| + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C \quad (8)$$

$$\int_2^{2\sqrt{3}} g(x)\,dx = \ln 2\sqrt{3} - \ln 2 + $$
$$+ \frac{1}{2}\left(\operatorname{arctg}\sqrt{3} - \operatorname{arctg} 1\right)$$

$$= \ln\sqrt{3} + \frac{1}{2}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \ln\sqrt{3} + \pi/24 \qquad (4)$$

4) a) $\det A = (k-1)^2(k+2)$ (4)

A is singular iff $k=1$ or $k=-2$ (3)

In this case, the rows are lin. dependent. (3)

b) $BX = A \iff X = B^{-1}A$ (2)

$$\iff X = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix} \qquad (8)$$

5 a) $\begin{cases} x + ky = 2 \\ kx + y = m \\ ky + z = 0 \end{cases} \iff \left[\begin{array}{ccc|c} 1 & k & 0 & 2 \\ k & 1 & 0 & m \\ 0 & k & 1 & 0 \end{array}\right]$ (4)

$\underbrace{}_{A}$

$\det A = 1 - k^2$ (7)

If $k \neq \pm 1$, then the system has a unique solution (4)

If $k = \pm 1$, the system may be undefined or impossible.

b) $k = 2$, $m = 1$

$\det A = 1 - 2^2 = -3.$ (9)

$$x = \frac{\det \begin{pmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}}{-3} = 0$$ (3)

$y = 1$ (Same rule) (3)

$z = -2$ (3)

Solution

$\boxed{\{(0, 1, -2)\}}$