Mathematics 2

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Lecture 1: Eigenvalues and Eigenvectors

Let $A \in M_{n\times n}(\mathbb{R})$; that is, a collection of n^2 numbers arranged in n rows and n columns

We consider the linear transformation

$$A : \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

determined by the matrix-vector product.

By *linear* we mean that it maintains the sum of vectors (u, v) and multiplication of scalars (b, c):

$$A(bu + cv) = bA(u) + cA(v) = bAu + cAv,$$

for all n-dimensional vectors $u, v \in \mathbb{R}^n$, and all b, c real numbers.

What is the relationship between v and Av?

For a general vector v, the vectors v and Av need **not have the same direction**.

This is due to the fact that, usually, multiplication by a matrix changes both the direction and length of a vector.

Eigenvalues and **eigenvectors** identify directions that are preserved by a linear transformation.

Definition 1

Let A be an $n \times n$ matrix. A scalar λ is an ***eigenvalue*** of A if there exists a vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v.$$

The vector v is called an ***eigenvector*** corresponding to λ.

Remark 2

Observe that the zero vector **provides no additional information** about A:

$$A0 = \lambda 0$$

for every scalar λ.

Hence, we shall impose the condition

$$\boxed{v \neq 0}$$

in the definition of eigenvector.

Example 3

Consider

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

- Notice that

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad Av_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

  In other words, multiplication by A re-scales v_1 by a factor of 1, and leaves its direction unchanged.

- Similarly,

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \Longrightarrow \qquad Av_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

  That is to say, multiplication by A re-scales v_2 by a factor of 3, and leaves its direction unchanged.

Suppose

$$\boxed{Av = \lambda v.}$$

- If $\lambda > 1$, the vector is stretched.
- If $0 < \lambda < 1$, the vector is contracted.
- If $\lambda < 0$, the direction is reversed.
- If $\lambda = 0$, the vector is mapped to the zero vector.

Suppose

$$\boxed{Av = \lambda v.}$$

For every scalar c,

$$A(cv) = cAv = c\lambda v = \lambda(cv).$$

Every nonzero scalar multiple of an eigenvector is again an eigenvector for the same eigenvalue.

Definition 4

For an eigenvalue λ of A, the ***eigenspace*** corresponding to λ is

$$E_\lambda = \{v \in \mathbb{R}^n : (A - \lambda I)v = 0\}.$$

Remark 5

From Definition 4, it is immediate that

$$E_\lambda = \ker(A - \lambda I).$$

Consequently, eigenspaces are subspaces of $\mathbb{R}^n$.

In particular:

- $0 \in E_\lambda$;
- sums of vectors in E_λ remain in E_λ;
- scalar multiples remain in E_λ.

The eigenvectors corresponding to λ are precisely the *nonzero* vectors in E_λ.

Finding Eigenvalues

Start from

$$\boxed{Av = \lambda v.}$$

Rearranging gives

$$(A - \lambda I)v = 0.$$

We want this system to have a nonzero solution.

Known fact: A homogeneous system has a nonzero solution precisely when its coefficient matrix is singular.

Thus, the eigenvalues of A are found by solving the ***characteristic equation***

$$\det(A - \lambda I) = 0.$$

Theorem 6

A scalar λ is an eigenvalue of A if and only if

$$\det(A - \lambda I) = 0.$$

Equivalently, define

$$p(\lambda) = \det(A - \lambda I).$$

The polynomial p is called the *characteristic polynomial* of A.

From Theorem 6, *the eigenvalues of A correspond the roots of p.*

As an algorithm: Given an $n \times n$ matrix A:

1. Form $A - \lambda I$.
2. Compute $\det(A - \lambda I)$.
3. Solve
$$\det(A - \lambda I) = 0.$$

The resulting values of λ are the eigenvalues.

Example 7 (Triangular matrix)

Consider

$$A = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}.$$

Then

$$A - \lambda I = \begin{pmatrix} 4 - \lambda & 1 \\ 0 & 2 - \lambda \end{pmatrix}.$$

Hence, the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = (4 - \lambda)(2 - \lambda).$$

Example 7 (cont.)

The characteristic equation is

$$(4-\lambda)(2-\lambda)=0.$$

Therefore, the eigenvalues of A are

$$\boxed{\lambda_1 = 4, \qquad \lambda_2 = 2.}$$

Finding Eigenvectors

Once an eigenvalue λ is known, solve

$$(A - \lambda I)v = 0.$$

This is an ordinary homogeneous linear system, and the **set of all of its solutions is the eigenspace E_λ.**

Example 7 (cont.)

We now look for eigenvectors corresponding to the eigenvalue $\lambda = 4$ from the matrix

$$A = \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix}.$$

In this case, we have

$$A - 4I = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}.$$

Then, the associated linear system is

$$\begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example 7 (cont.)

The equations give

$$y = 0.$$

Hence

$$E_4 = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

Therefore

$$E_4 = \operatorname{span} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Suggested Exercise: find the eigenvectors and eigenspace of $\lambda = 2$.

Definition 8

Let A be a matrix, and let λ be an eigenvalue of A.

1. The *geometric multiplicity* of λ is defined as the dimension of the corresponding eigenspace E_λ:

   $$\mathrm{gm}(\lambda) = \dim E_\lambda.$$

   I.e., it is the number of linearly independent eigenvectors that can be chosen for λ.

2. The *algebraic multiplicity* of λ is defined as its multiplicity as a root of the characteristic polynomial $p(\lambda)$.

Proposition 9

For every eigenvalue λ of A,

$$1 \leq \mathrm{gm}(\lambda) \leq \mathrm{am}(\lambda).$$

Suggested Exercise: Consider

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

What are the eigenvalues of A? What are their multiplicities?

Invertibility

Of particular interest is the case of $\lambda = 0$. Indeed: in this case the characteristic equation

$$Av = 0v = 0$$

has a nonzero solution, which implies that A has a nontrivial kernel.

Therefore

$$0 \text{ is an eigenvalue of } A \iff A \text{ is singular.}$$

Theorem 10

For an $n \times n$ matrix A, the following are equivalent:

1. *0 is an eigenvalue of A,*
2. $\det(A) = 0$,
3. *A is singular,*
4. *A is **not** invertible.*

Diagonalisation

A fundamental property is that eigenvectors corresponding to distinct eigenvalues are linearly independent.

Theorem 11

Let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of A, with corresponding eigenvectors $v_1, \ldots, v_k$. Then

$$v_1, \ldots, v_k$$

are linearly independent.

Proof.

Blackboard. $\square$

Corollary 12

If an $n \times n$ matrix has n distinct eigenvalues, then it has n linearly independent eigenvectors.

Consequently, these eigenvectors form a basis of $\mathbb{R}^n$.

Proof (Sketch).

Suppose

$$v_1, \ldots, v_n$$

are linearly independent eigenvectors of A, with

$$Av_i = \lambda_i v_i.$$

Let

$$P = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Then

$$AP = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

If P is invertible, then

$$A = PDP^{-1},$$

where

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Thus eigenvectors give a basis in which the matrix becomes diagonal. $\square$

Suggested exercise: Let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find:

1. the characteristic polynomial;
2. the eigenvalues;
3. the algebraic multiplicity of each eigenvalue;
4. the eigenspace corresponding to each eigenvalue.

Is the matrix diagnosable?

Summary

- An eigenvalue λ and eigenvector $v \neq 0$ satisfy
$$\boxed{Av = \lambda v.}$$
- Eigenvalues are found using the characteristic equation.
- Eigenvectors are found by solving
$$(A - \lambda I)v = 0.$$
- The eigenspace is
$$E_\lambda = \ker(A - \lambda I).$$
- Geometric multiplicity is $\dim E_\lambda$.
- Algebraic multiplicity is multiplicity as a root of the characteristic polynomial.
- Distinct eigenvalues have linearly independent eigenvectors.
- 0 is an eigenvalue $\iff$ A is singular.
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.
- n distinct eigenvalues of an $n \times n$ matrix give a basis of eigenvectors.
- A basis of eigenvectors leads to a diagonal representation of A.

References

[1] Gene H. Golub and Charles F. Van Loan. *Matrix computations*. eng. 6. print. Johns Hopkins series in the mathematical sciences 3. Baltimore, Md: Johns Hopkins Univ. Pr, 1988.

[2] Serge Lang. *Linear Algebra*. en. Ed. by S. Axler, F. W. Gehring, and K. A. Ribet. Undergraduate Texts in Mathematics. New York, NY: Springer New York, 1987. DOI: 10.1007/978-1-4757-1949-9.