Mathematics 2

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Lecture 2: Quadratic Forms[1]

[1] Some additional references: Details of the theory can be found in [2]; numerical/computational aspects are discussed with further detail in [1] (in the library).

A **quadratic form** is a homogeneous polynomial of degree two in several variables.

Quadratic forms arise naturally in **geometry**, **optimisation**, **statistics**, and differential equations.

Definition 1

A ***quadratic form in two variables*** is a function

$$Q : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

of the form

$$Q(x, y) = a_{11}x^2 + a_{12}xy + a_{21}yx + a_{22}y^2.$$

Observe: Since $xy = yx$, we may combine the two mixed terms:

$$Q(x, y) = a_{11}x^2 + \underbrace{(a_{12} + a_{21})}_{=b_{12}} xy + a_{22}y^2.$$

More generally, a quadratic form in **two variables** can be written as

$$Q(x, y) = ax^2 + bxy + cy^2.$$

Example 2

Consider

$$Q(x, y) = 5x^2 + 8xy + 7y^2.$$

This is a quadratic form in $\mathbb{R}^2$, where

$$a_{11} = 5, \qquad b_{12} = (a_{12} + a_{21}) = 8, \qquad a_{22} = 7.$$

The important point: every term has total degree 2.

Definition 3

A ***quadratic form in three variables*** is a function

$$Q : \mathbb{R}^3 \to \mathbb{R}$$

of the form

$$\begin{aligned} Q(x,y,z) = \quad & a_{11}\, x^2 + a_{12}\, xy + a_{13}\, xz \\ + \; & a_{21}\, yx + a_{22}\, y^2 + a_{23}\, yz \\ + \; & a_{31}\, zx + a_{32}\, zy + a_{33}\, z^2. \end{aligned} \tag{1}$$

Observe: Collecting the mixed terms, (1) is equivalent to

$$
\begin{aligned}
Q(x,y,z) = \quad & a_{11}\, x^2 + b_{12}\, xy + b_{13}\, xz \\
& \qquad\quad + a_{22}\, y^2 + b_{23}\, yz \\
& \qquad\qquad\qquad + a_{33}\, z^2
\end{aligned}
$$

$$
\begin{aligned}
= \quad & a_{11}\, x^2 + \frac{1}{2}b_{12}\, xy + \frac{1}{2}b_{13}\, xz \\
& + \frac{1}{2}b_{12}\, yx + a_{22}\, y^2 + \frac{1}{2}b_{23}\, yz \\
& + \frac{1}{2}b_{13}\, zx + \frac{1}{2}b_{23}\, zy + a_{33}\, z^2.
\end{aligned}
$$

Definition 4

A ***quadratic form in*** n ***variables*** is a function

$$Q : \mathbb{R}^n \longrightarrow \mathbb{R}$$

of the form

$$Q(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j, \tag{2}$$

where $a_{11}, a_{12}, \ldots, a_{nn}$ are some given (fixed) real numbers, here-on-out referred as *coefficients*.

Observe: As in the 2− and 3−dimensional cases, (2) is equivalent to

$$Q(x_1, \ldots, x_n) = \sum_{1 \leq i \leq j \leq n} b_{ij} x_i x_j, \tag{3}$$

for some coefficients $b_{11}, b_{12}, \ldots, b_{nn}$.[2]

It is precisely this structure that allows a **matrix representation**.

[2] It should be clear that the coefficients a_{ij} and b_{ij} need not be the same.

Proposition 5

Every quadratic form Q in $\mathbb{R}^n$ can be represented using a square matrix. More precisely:

$$\boxed{Q(x) = x^\top A x}$$

for all $x \in \mathbb{R}^n$.

Furthermore: without loss of generality, the matrix can be taken symmetric (or triangular).

Idea of the proof.

1. Let
$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad A = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}.$$
2. Using the definition of vector-matrix and matrix-vector multiplication, compute $x^\top Ax$.
3. Verify that the product $x^\top Ax$ coincides with (3).
4. Corroborate the equivalence between (2) and (3), and conclude how the symmetric matrix representation is obtained.

$\square$

Corollary 6

*The symmetric matrix associated with a quadratic form is **unique**.*

Proof.

Suggested exercise. $\square$

Example 7

Let

$$Q(x, y) = 5x^2 + 8xy + 7y^2.$$

Then, from the definition of matrix multiplication:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 8 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5x^2 + 8xy + 7y^2.$$

Alternatively:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 5x^2 + 8xy + 7y^2.$$

Important observation: Having the same product **DOES NOT IMPLY** that the matrices are the same.

Example 8

Consider

$$Q(x,y,z) = 4x^2 + 8xy + 12xz + 8y^2 + 5yz + 20z^2.$$

Its associated symmetric matrix is

$$A = \begin{pmatrix} 4 & 4 & 6 \\ 4 & 8 & \frac{5}{2} \\ 6 & \frac{5}{2} & 20 \end{pmatrix}.$$

Therefore

$$\boxed{Q(x,y,z) = \begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix}.}$$

Some intuition: In one dimension, the function

$$f(x) = ax^2$$

corresponds to a parabola.

Then, exactly one of the following conditions hold:

- $a > 0 \implies$ the parabola opens upwards
  $\implies$ the function f can be *minimized*.
- $a < 0 \implies$ the parabola opens downwards
  $\implies$ the function f can be *maximized*.
- $a = 0 \implies$ the parabola is flat
  $\implies$ the function f is constant; i.e. there is nothing to *minimize* or *maximize*.

In the multidimensional case, we now have expressions of the form

$$\boxed{Q(x) = x^\top Ax.}$$

Building up from our intuition, how can we define the *sign of the matrix* A?

In particular, we interested in the following:

- Is $Q(x)$ always positive?
- Is $Q(x)$ always negative?
- Can $Q(x)$ be zero for some $x \neq 0$?
- Can $Q(x)$ take both positive and negative values?

These questions lead to the **classification of quadratic forms**.

Remark 9

Since every quadratic form has a **unique symmetric matrix** representing (Corollary 6), when classifying a quadratic form we will always work with its associated symmetric matrix.

Definition 10

Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form.

- Q is ***positive definite*** if

$$Q(x) > 0 \qquad \text{for all } x \neq 0.$$

- Q is ***positive semi-definite*** if

$$Q(x) \geq 0 \qquad \text{for all } x,$$

  and $Q(y) = 0$ for some $y \neq 0$.

Definition 10 (cont.)

- Q is ***negative definite*** if

$$Q(x) < 0 \qquad \text{for all } x \neq 0.$$

- Q is ***negative semi-definite*** if

$$Q(x) \leq 0 \qquad \text{for all } x,$$

  and $Q(y) = 0$ for some $y \neq 0$.
- Q is ***indefinite*** if there exist x, y such that

$$Q(x) > 0 \qquad \text{and} \qquad Q(y) < 0.$$

Example 11

Consider

$$Q(x, y, z) = -3x^2 - 2y^2 - z^2.$$

For every $(x, y, z) \neq (0, 0, 0)$,

$$Q(x, y, z) < 0.$$

Hence

$$\boxed{Q \text{ is negative definite.}}$$

Example 12

Consider

$$Q(x, y, z) = x^2 - 2xz + y^2 + z^2 = (x - z)^2 + y^2.$$

Hence $Q \geq 0$, but

$$Q(1, 0, 1) = 0.$$

Therefore Q is positive semi-definite.

Example 13

Consider

$$Q(x, y, z) = x^2 - 2xz - y^2 + z^2.$$

Evaluate Q at two different points:

$$Q(0, 1, 0) = -1 < 0,$$

while

$$Q(2, 0, 0) = 4 > 0.$$

Thus Q takes both positive and negative values.

$$\boxed{Q \text{ is indefinite.}}$$

Another way to classify quadratic forms is through the information provided by the eigenvalues of the corresponding symmetric matrix. This is particularly useful for complicated quadratic forms where direct evaluation can be impractical.

Remark 14

In what follows, we say the *matrix* A is *positive definite* (respectively *negative definite*, *indefinite*, etcetera), if the *quadratic form* $Q(x) = x^\top Ax$ *positive definite* (respectively *negative definite*, *indefinite*, etcetera).

Proposition 15

Let A be a symmetric matrix.

1. *A is positive definite $\iff$ all eigenvalues are positive.*
2. *A is positive semi-definite $\iff$ at least one eigenvalue is zero and all the others are non-negative.*
3. *A is negative definite $\iff$ all eigenvalues are negative.*
4. *A is negative semi-definite $\iff$ at least one eigenvalue is zero and all the others are non-positive.*
5. *A is indefinite $\iff$ A has at least one positive and one negative eigenvalue.*

Eigenvalues of A	**Classification of** $Q(x) = x^\top Ax$
$\lambda_i > 0$ for all i	Positive definite
$\lambda_i \geq 0$ for all i, $\lambda_i = 0$ for some i	Positive semi-definite
$\lambda_i < 0$ for all i	Negative definite
$\lambda_i \leq 0$ for all i, $\lambda_i = 0$ for some i	Negative semi-definite
$\lambda_i > 0$ and $\lambda_j < 0$	Indefinite

Example 16

Consider

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since A is diagonal, its eigenvalues are

$$\lambda_1 = -3, \qquad \lambda_2 = -2, \qquad \lambda_3 = -1.$$

All eigenvalues are negative.

$$\boxed{Q(x) = x^\top Ax \text{ is negative definite.}}$$

Example 17

Consider

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = 0, \qquad \lambda_3 = 2.$$

All eigenvalues are non-negative, and one is zero.

$$\boxed{Q(x) = x^\top Ax \text{ is positive semi-definite.}}$$

Example 18

Consider

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

The eigenvalues are

$$\lambda_1 = -1, \qquad \lambda_2 = 0, \qquad \lambda_3 = 2.$$

There is both a positive and a negative eigenvalue.

$$\boxed{Q(x) = x^\top A x \text{ is indefinite.}}$$

The last method we present for the classification of quadratic forms/symmetric matrices is based on **principal minors**. This method can prove useful whenever computing eigenvalues becomes difficult, particularly for matrices of large order.

Definition 19

Let A be an $n \times n$ symmetric matrix. The ***leading principal submatrix of order*** k is obtained by deleting the last $n - k$ rows and columns.

Example 20

Consider the $n \times n$ matrix.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} & a_{1k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} & a_{2k+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} & a_{k-1k+1} & \cdots & a_{k-1n} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} & a_{kk+1} & \cdots & a_{kn} \\ a_{k+11} & a_{k+12} & \cdots & a_{k+1k-1} & a_{k+1k} & a_{k+1k+1} & \cdots & a_{k+1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk-1} & a_{nk} & a_{nk+1} & \cdots & a_{nn} \end{pmatrix}$$

Example 20 (cont.)

The principal submatrix of order k is obtained from the 'first' k^2 entries (i.e. the elements in the first k rows/columns):

$$\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} & a_{1k+1} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} & a_{2k+1} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} & a_{k-1k+1} & \cdots & a_{k-1n} \\
a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} & a_{kk+1} & \cdots & a_{kn} \\
a_{k+11} & a_{k+12} & \cdots & a_{k+1k-1} & a_{k+1k} & a_{k+1k+1} & \cdots & a_{k+1n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nk-1} & a_{nk} & a_{nk+1} & \cdots & a_{nn}
\end{pmatrix}$$

Example 20 (cont.)

The rest of the rows/columns are discarded:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{k-11} & a_{k-12} & \cdots & a_{k-1k-1} & a_{k-1k} \\ a_{k1} & a_{k2} & \cdots & a_{kk-1} & a_{kk} \end{pmatrix}$$

Definition 21

Let A be a $n \times n$ symmetric matrix. For every $k = 1, \ldots, n$, the k-***th principal minor of*** A is defined as the *determinant of the leading principal submatrix of order* k:

$$\Delta_k = \det \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}.$$

Theorem 22 (Sylvester's Criterion)

Let A be an $n \times n$ symmetric matrix with $\det A \neq 0$.

1. *A is positive definite if, and only if,*

$$\Delta_k > 0, \qquad k = 1, \ldots, n.$$

2. *A is negative definite if, and only if,*

$$(-1)^k \Delta_k > 0, \qquad k = 1, \ldots, n.$$

Remark 23

Sylvester's criterion does not classify matrices for which

$$\det A = 0.$$

In other words, the method of minors does not classify

- positive semi-definite quadratic forms;
- negative semi-definite quadratic forms;
- indefinite quadratic forms.

Summary

- A quadratic form is a homogeneous polynomial of degree two.
- In $\mathbb{R}^n$,
$$Q(x) = x^\top Ax.$$
- Every quadratic form has a unique **symmetric matrix** representation.
- A quadratic form is classified as positive definite, positive semi-definite, negative definite, negative semi-definite, or indefinite.
- Classification can be performed directly from the definition.
- For a symmetric matrix, the signs of its eigenvalues determine the classification.
- Positive definite $\iff$ all eigenvalues are positive.
- Negative definite $\iff$ all eigenvalues are negative.
- Indefinite $\iff$ there are both positive and negative eigenvalues.
- Sylvester's criterion uses leading principal minors to test positive and negative definiteness when $\det(A) \neq 0$.

References

[1] Gene H. Golub and Charles F. Van Loan. *Matrix computations*. eng. 6. print. Johns Hopkins series in the mathematical sciences 3. Baltimore, Md: Johns Hopkins Univ. Pr, 1988.

[2] Serge Lang. *Linear Algebra*. en. Ed. by S. Axler, F. W. Gehring, and K. A. Ribet. Undergraduate Texts in Mathematics. New York, NY: Springer New York, 1987. DOI: 10.1007/978-1-4757-1949-9.